Chapter 6
State-Space Design

Two steps

1. Assumption is made that we have all the states at our disposal for feedback purposes (in practice, we would not measure all these states). This allows us to implement a control law.

2. Estimator or observer design

The dynamic system obtained from the continued control law and estimator is called the **controller**

Control law

Consider a linear combination of the states

$$u = -Kx = -\begin{bmatrix} k_1 & k_2 & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

This control law assumes that $r = 0$ and usually referred to as a **regulator**.

We have the difference equation

$$x(k+1) = \phi x(k) + \Gamma u(k)$$

Substituting u=- K x $\Rightarrow$

$$x(k+1) = \phi x(k) - \Gamma K x(k)$$

Taking z transform

$$(zI - \phi + \Gamma K)X(z) = 0$$

The characteristic equation is

$$det(zI - \phi + \Gamma K) = 0$$

Pole Placement

Given desired root locations

$$z_i = \beta_1,\ \beta_2,\ \beta_3,\ \cdots$$

the desired characteristic equation is

$$\alpha_c(z) = (z - \beta_1)\ (z - \beta_2)\ (z - \beta_3)\cdots = 0$$

Example

Design a control law for the plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{F} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{G} u$$

The discrete model of the system is

$$\phi = e^{FT}, \qquad \Gamma = \int_0^T e^{F\eta}\, d\eta\, G$$

$$\Rightarrow \qquad \phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$$

Suppose we wish to pick z-plane roots of the closed-loop characteristic equation so that equivalent s-plane roots have a damping ratio of $\zeta = 0.5$ and a real part of $s = -1.8 rad/sec$ ($i.e.\ s = -1.8 \pm j3.12$)

Using $z = e^{sT}$ with $T = 0.1$ we find

$$z = 0.8 \pm j0.25$$

$\Rightarrow$ the desired characteristic equation is

$$z^2 - 1.6z + 0.70 = 0 \tag{1}$$

$$\Rightarrow \qquad det\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) = 0$$

or

$$z^2 + \left(Tk_2 + \left(\frac{T^2}{2}\right) k_1 - 2 \right) z + \left(\frac{T^2}{2}\right) k_1 - Tk_2 + 1 = 0 \tag{2}$$

equating coefficients of like power in (1) and (2)

$$Tk_2 + \left(\frac{T^2}{2}\right) k_1 - 2 = -1.6$$

$$\left(\frac{T^2}{2}\right) k_1 - Tk_2 + 1 = 0.70$$

$$\Rightarrow \qquad k_1 = \frac{0.10}{T^2} = 10, \qquad k_2 = \frac{0.35}{T} = 3.5, \qquad for\ T = 0.1$$

The above calculations are easier if use is made of the *control canonical form*

$$\Phi_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \Gamma_c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad H_c = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

The characteristic equation of Φ_c is

$$a(z) = z^3 + a_1 z^2 + a_2 z + a_3$$

i.e. the coefficients of the characteristic polynomial are the negative of the coefficients of the first row of Φ_c.
The closed-loop system matrix $\Phi_c - \Gamma_c K$ is

$$\Phi_c - \Gamma_c K = \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$z^3 + (a_1 + k_1) z^2 + (a_2 + k_2) z + (a_3 + k_3) = 0$$

If the desired root locations are given by the roots of the equation

$$z^3 + \alpha_1 z^2 + \alpha_2 z + \alpha_3 = 0$$

then the gains are

$$k_1 = \alpha_1 - a_1 \ ; \qquad k_2 = \alpha_2 - a_2 \ ; \qquad k_3 = \alpha_3 - a_3$$

Steps in canonical-form design method

Given an arbitrary $(\Phi, \ \Gamma)$ and desired characteristic equation $\alpha(z) = 0$, by redefining the states we convert $(\Phi, \ \Gamma)$ to control form $(\Phi_c, \ \Gamma_c)$ and find the gain as shown above.
Then we transform the gains back in terms of the original states.

<u>Controllability</u>

Is it always possible to find an equivalent $(\Phi_c, \ \Gamma_c)$ for arbitrary $(\Phi, \ \Gamma)$?

If the roots of $det(zI - \phi) = 0$ are distinct, then the state equation may be transformed into the following form

$$x(k+1) = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x(k) + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_n \end{bmatrix} u(k)$$

Criterion for controllability: no element of Γ can be zero.

Ackerman's Formula

Given that $(\Phi_c,\ \Gamma_c)$ exists, i.e.the system is controllable, then the gains to implement a control law are given by

$$K = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}^{-1} \alpha_c(\Phi)$$

where $C = \begin{bmatrix} \Gamma & \Phi\Gamma \cdots \end{bmatrix}$ is the controllability matrix, n is the order of the system and we substitute Φ for z in $\alpha_c(z)$ to form

$$\alpha_c(\Phi) = \Phi^n + \alpha_1 \Phi^{n-1} + \alpha_2 \Phi^{n-2} + \cdots + \alpha_n I$$

where $\alpha_i's$ are the coefficients of the desired characteristic equation.

$$\alpha_i(z) = |zI - \phi + \Gamma k| = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n$$

Example

Redo last example

$$\Rightarrow \qquad \alpha_1 = -1.6, \qquad \alpha_2 = 0.70$$

Since desired characteristic equation is $z^2 - 1.6z + 0.70 = 0$

$\Rightarrow$

$$\alpha_c(\Phi) = \begin{bmatrix} 1 & 2T \\ 0 & 1 \end{bmatrix} - 1.6 \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + 0.7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix}$$

also

$$\begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} \frac{T^2}{2} & \frac{3T^2}{2} \\ T & T \end{bmatrix}$$

and

$$\begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix}^{-1} = \frac{1}{T^2} \begin{bmatrix} -1 & \frac{3T}{2} \\ 1 & \frac{-T}{2} \end{bmatrix}$$

$$\Rightarrow \qquad K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \frac{1}{T^2} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & \frac{3T}{2} \\ 1 & \frac{-T}{2} \end{bmatrix} \begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \frac{1}{T^2} \begin{bmatrix} 0.1 & 0.35T \end{bmatrix} = \begin{bmatrix} 10 & 3.5 \end{bmatrix}$$

Which is the same result as before

Estimator Design

There are two kinds of basic estimates of the state x(k):

1. The current estimate, $\hat{x}(k)$ is based on measurements y(k) up to and including the kth instant.

2. predictor estimate, $\bar{x}(k)$ is based on measurements up to $y(k-1)$

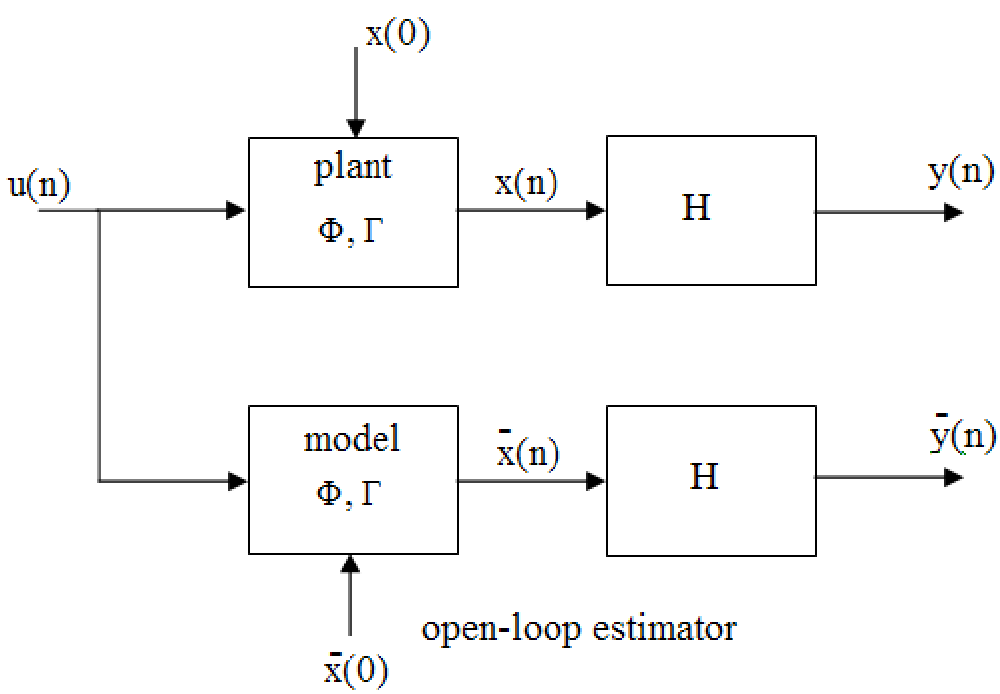
We will like to set

$$u = -K\hat{x} \qquad or \qquad u = -K\bar{x}$$

Prediction estimator

To estimate the state we could construct a model of the plant dynamics

$$\bar{x}(k+1) = \Phi\bar{x}(k) + \Gamma u(k)$$



We know Φ, Γ and $u(k)$, so the above system should work if we can obtain x(0) and set $\bar{x}(0)$ equal to it.

Define an error in the estimate

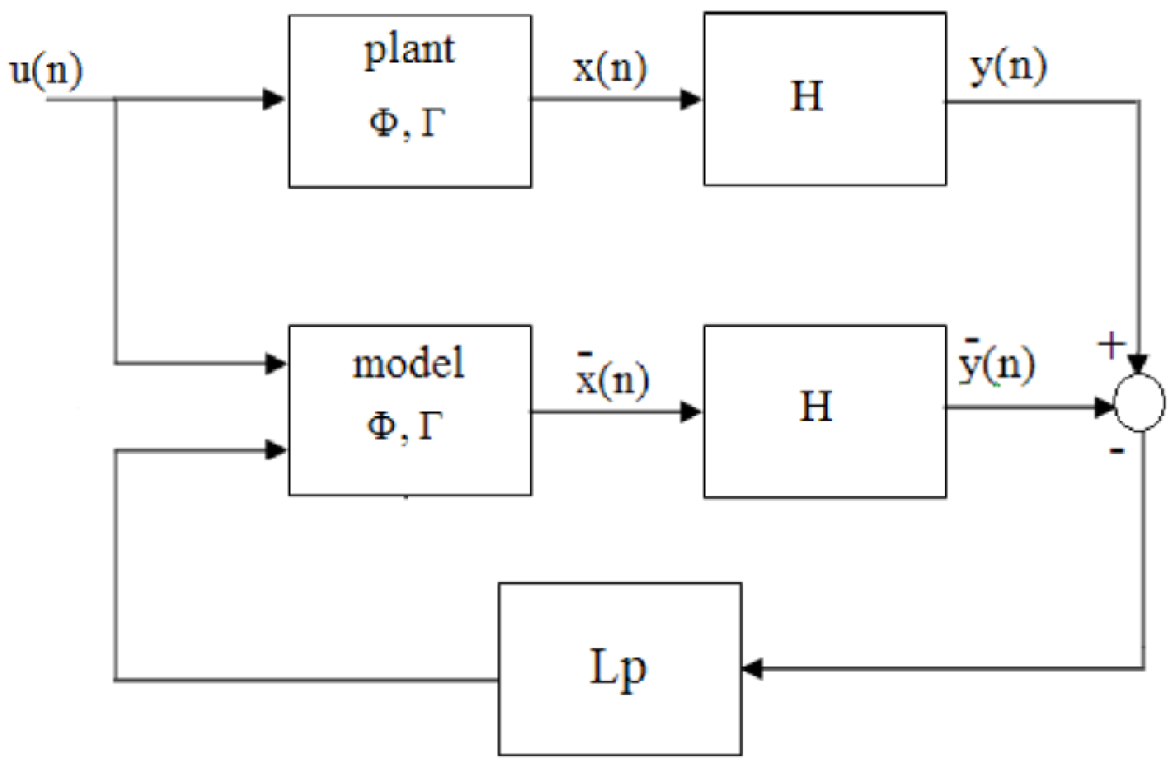
$$\tilde{x} = \bar{x} - x \qquad (*)$$

The dynamics of the estimator-error is given by

$$\tilde{x}(k+1) = \Phi\tilde{x}(k)$$

if the initial value of $\bar{x}$ is off, the dynamics are those of the uncompensated plant, Φ.

The estimator is running open-loop and not utilizing any continuing measurements of the system's behavior, and so we expect it to diverge from the truth. However, we can incorporate feedback to better the estimates.



Closed-loop Estimator

The closed-loop estimator is described by

$$\bar{x}(k+1) = \Phi\bar{x}(k) + \Gamma u(k) + L_p[y(k) - H\ \bar{x}(k)] \qquad (**)$$

where L_p is the feedback gain matrix This is a *Prediction estimator* because a measurement at time k results in an estimate of the state that is valid at time $k+1$, i.e. the estimate has been predicted one cycle in the future.

The dynamics of the estimator error is found by subtracting

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Hx(k) \qquad from(**) \\ \tilde{x}(k+1) &= [\Phi - L_p H]\tilde{x}(k) \end{aligned}$$

Due to feedback action, $\bar{x}(k)$ will converge to $x(k)$ regardless of $\bar{x}(0)$ and will do so quickly particularly if L_p is large (assuming a stable system matrix).

To find L_p, we do the same as for designing the control law. Specify the desired estimator root locations in the z-plane

$$char.\ eqn.\ (z-\beta_1)(z-\beta_2)\cdots(z-\beta_n) = 0 \qquad (\dagger)$$

equate the coefficients of $(\dagger)$ and

$$|zI - \Phi + L_p H| = 0$$

Example

Construct estimator for previous examples. The measurement is of x_1 $\Rightarrow H = [1 \quad 0]$. The desired roots are $z = 0.4 \pm j0.4 \;\Rightarrow$ s-plane roots with $\zeta = 0.6$ *and* ω_n that is 3 times faster than the control roots selected.

Desired characteristic equation gives (approx.)

$$z^2 - 0.8z + 0.32 = 0$$

$$|zI - \Phi + L_p H| = 0 \qquad \Rightarrow$$

$$det\left(z\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix} + \begin{bmatrix}L_{p1}\\ L_{p2}\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right) = 0$$

or

$$z^2 + (L_{p1} - 2)z + TL_{p2} + 1 - L_{p1} = 0$$

$$\Rightarrow \qquad L_{P1} = 1.2$$

$$L_{P2} = \frac{0.52}{T} = 5.2$$

The estimator algorithm to be coded in the computer is

$$\bar{x}_1(k+1) = \bar{x}_1(k) + 0.005u(k) + 0.1\bar{x}_2(k) + 1.2[y(k) - \bar{x}_1(k)]$$

$$\bar{x}_2(k+1) = \bar{x}_2(k) + 0.1u(k) + 5.2[y(k) - \bar{x}_1(k)]$$

Since

$$\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$$

$T = 0.1$

$$\Phi = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix}$$

<u>Observability</u>

Given a desired set of estimator roots, is L_p uniquely determined? It is provided y is a scalar and the system is "observable".

<u>Ackerman's Formula</u>

L_p may be determined from

$$L_p = \alpha_e(\Phi) \underbrace{\begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \\ \vdots \\ H\phi^{n-1} \end{bmatrix}^{-1}}_{\mathcal{O}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where

$$\alpha_e(\Phi) = \Phi^n + \alpha_1\Phi^{n-1} + \alpha_2\Phi^{n-2} + \cdots + \alpha_n I$$

and the $\alpha_i' s$ are the coefficients of the desired characteristic equation, i.e.

$$\alpha_e(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n$$

$\mathcal{O}$ is the *observability matrix* and it must be full rank for the matrix to be invertible and for the system to be observable.

Current Estimator

For the prediction estimator, the state estimate $\bar{x}(k)$ is arrived at after receiving measurements up through $y(k-1)$. The current estimator, estimates the state $\hat{x}(k)$ based on the current measurement y(k).

$$\hat{x}(k) = \bar{x}(k) + L_c(y(k) - H\bar{x}(k)) \tag{3}$$

where $\bar{x}(k)$ is the predicted estimate based on a model prediction from the previous time estimate, that is

$$\bar{x}(k) = \Phi\hat{x}(k-1) + \Gamma u(k-1) \tag{4}$$

(3) and (4) $\Rightarrow$

$$\bar{x}(k+1) = \Phi\bar{x}(k) + \Gamma u(k) + \Phi L_c[y(k) - H\bar{x}(k)] \tag{5}$$

The estimation-error equation for $\bar{x}(k)$ is

$$\tilde{x}(k+1) = [\Phi - \Phi L_c H]\tilde{x}(k) \tag{6}$$

where $\tilde{x} = \bar{x} - x$
From the above we see that $\bar{x}$ in the current estimator equation is the same quantity as $\bar{x}$ in the predictor estimator equation and the estimator gain matrices are related by

$$L_p = \Phi L_c \tag{7}$$

the estimator-error equation for $\hat{x}$ is

$$\tilde{x}(k+1) = [\Phi - L_c H\Phi]\tilde{x}(k) \tag{8}$$

Where $\tilde{x} = \hat{x} - x$

(6) and (8) can be shown to have the same roots. Therefore one could use either form as the basis for computing the estimator gain, L_c.

Using (8) which is similar to prediction estimator result except that $H\Phi$ appears instead of H, Ackerman's formula for L_c is

$$L_c = \alpha_e(\Phi)\begin{bmatrix} H\Phi \\ H\Phi^2 \\ H\Phi^3 \\ \vdots \\ H\phi^n \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{9}$$

where $\alpha_e(\Phi)$ is based on the desired root locations.
Note (7) $\Rightarrow$

$$L_c = \Phi^{-1}L_p \tag{10}$$

Example

Repeat previous example using the current estimator formulation

$$(8) \qquad \Rightarrow \qquad {L_c}^T = [0.68 \qquad 5.2]$$

The estimator implementation using (3) and (4) in a way that reduces the computation delay as much as possible is, before sampling

$$\bar{x}_1(k) = \hat{x}_1(k-1) + 0.005u(k-1) + 0.1\hat{x}_2(k-1)$$

$$\bar{x}_2(k) = \hat{x}_2(k-1) + 0.1u(k-1)$$
$$x_1' = (1-0.68)\bar{x}_1(k)$$
$$x_2' = \bar{x}_2(k) - 5.2\bar{x}_1(k)$$

and after sampling y(k)

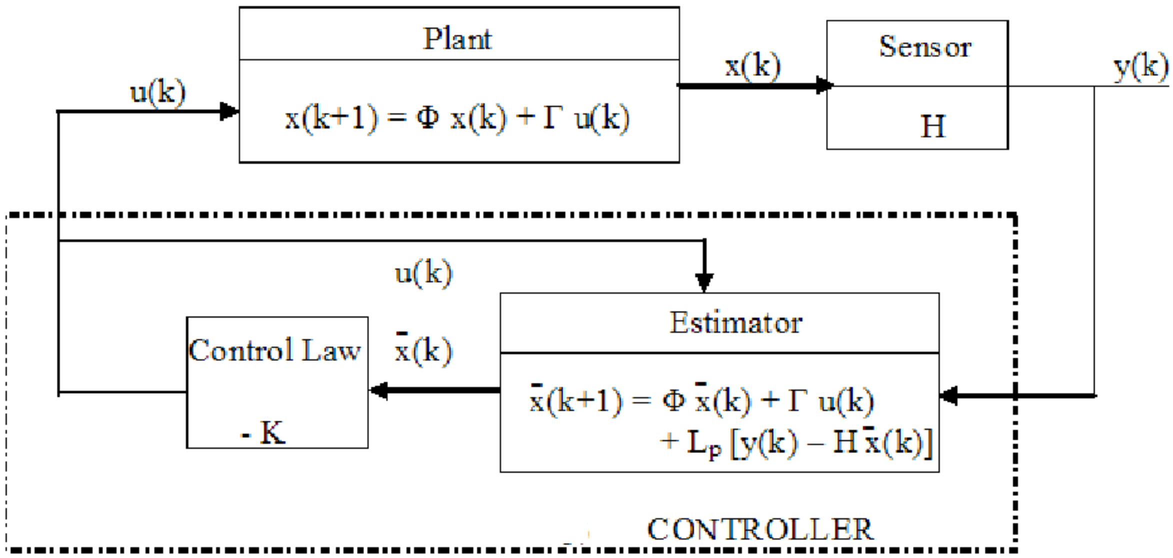
$$\hat{x}_1(k) = x_1' + 0.68y(k)$$

$$\hat{x}_2(k) = x_2' + 5.2y(k)$$

Reduced-Order Estimator

Not enough time to cover it.

Regulator Design: Combined Control Law and Estimator



The Separation Principle

Setting $\quad u = -K\bar{x}$

$$
\begin{aligned}
x(k+1) &= \Phi x(k) - \Gamma K \bar{x}(k) \\
&= \Phi x(k) - \Gamma K (x(k) + \tilde{x}(k))
\end{aligned}
$$

Combining this with the estimator-error equation gives

$$
\begin{bmatrix} \tilde{x}(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} \Phi - L_p H & 0 \\ -\Gamma K & \Phi - \Gamma K \end{bmatrix} \begin{bmatrix} \tilde{x}(k) \\ x(k) \end{bmatrix}
$$

The characteristic equation is

$$
\begin{vmatrix} zI - \Phi + L_p H & 0 \\ -\Gamma K & zI - \Phi + \Gamma K \end{vmatrix} = 0
$$

$$
\Rightarrow \qquad |zI - \Phi + L_p H||zI - \Phi + \Gamma K| = \alpha_e(z)\alpha_c(z) = 0
$$

i.e. the characteristic roots of the complete system consist of the combination of the estimator roots and the control roots that are unchanged from those obtained assuming actual state feedback.

It is interesting to compare the results of the "state-space designed compensator" to that of classical compensation.

Prediction estimator

$$\bar{x}(k) = (\Phi - \Gamma K - L_p H)\bar{x}(k-1) + L_p y(k-1)$$

$$u(k) = -K\bar{x}(k)$$

Current estimator

$$\hat{x}(k) = (\Phi - \Gamma K - L_c H\Phi + L_c H\Gamma K)\hat{x}(k-1) + L_c y(k)$$

$$u(k) = -K\hat{x}(k)$$

The poles of the controllers above are obtained from

$$|zI - \Phi + \Gamma K + L_p H| = 0$$

or

$$|zI - \Phi + \Gamma K + L_c H\Phi - L_c H\ \Gamma K| = 0$$

and are neither the control law poles nor the estimator poles.

Converting the difference equations to transfer functions results in

Prediction estimator

$$\frac{U(z)}{Y(z)} = D_p(z) = -K[zI - \Phi + \Gamma K + L_p H]^{-1} L_p$$

Current estimator

$$\frac{U(z)}{Y(z)} = D_c(z) = -K[zI - \Phi + \Gamma K + L_c H\Phi - L_c H\Gamma K]^{-1} L_c z$$