Chapter 5
State Space

Continuous-time

$$\dot{x} = Fx + Gu \tag{1}$$

$$y = H\ x \tag{2}$$

The solution of (1) is

$$x(t) = e^{F(t-t_0)}\ x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)}\ Gu(\tau)d\tau$$

the solution over one sample period is obtained by setting $t = kT + T$ *and* $t_0 = kT$

$$\Rightarrow \qquad x(kT+T) = e^{FT}x(kT) + \int_{kT}^{kT+T} e^{F(kT+T-\tau)}\ Gu(\tau)d\tau$$

Perform a change of variable in the integral from τ *to* η such that

$$\eta = kT + T - \tau$$

$$\Rightarrow \qquad x(kT+T) = e^{FT}x(kT) + \int_{0}^{T} e^{F\eta}\ d\eta\ Gu(kT)$$

where we have also assumed a ZOH on the input so that

$$u(\tau) = u(kT), \qquad kT \le \tau < kT + T$$

The final difference equations are

$$x(k+1) = \phi x(k) + \Gamma u(k)$$

$$y(k) = H\ x(k)$$

where

$$\phi = e^{FT}$$

$$\Gamma = \int_0^T e^{F\eta}\ d\eta G$$

Note that

$$\phi = e^{FT} = I + FT + \frac{F^2T^2}{2!} + \frac{F^3T^3}{3!} + \cdots$$

$\nwarrow$ state transition matrix

The discrete-time state equations are thus

$$x(k+1) = \phi x(k) + \Gamma u(k)$$

$$y(k) = H\ x(k)$$

Solution by recursion

$x(1) = \phi x(0) + \Gamma u(0)$
$x(2) = \phi x(1) + \Gamma u(1) = \phi^2 x(0) + \phi\Gamma u(0) + \Gamma u(1)$
$x(3) = \phi x(2) + \Gamma u(2) = \phi^3 x(0) + \phi^2\Gamma u(0) + \phi\Gamma u(1) + \Gamma u(2)$
$\vdots$
Repeating, we obtain

$$x(k) = \underbrace{\phi^k x(0)}_{contribution\ due\ to\ initial\ condition} + \underbrace{\sum_{j=0}^{k-1} \phi^{k-j-1}\Gamma u(j), \qquad k = 1,\ 2,\ 3, \cdots}_{contribution\ due\ to\ the\ input}$$

The output is

$$y(k) = H\phi^k x(0) + H\sum_{j=0}^{k-1} \phi^{k-j-1}\Gamma u(j)$$

Let us write the solution in terms of the state transition matrix

$$\psi(k) = \phi^k$$

$\Rightarrow$

$$\begin{aligned} x(k) &= \psi(k)\ x(0) + \sum_{j=0}^{k-1} \psi(k-j-1)\ \Gamma u(j) \\ &= \psi(k)\ x(0) + \sum_{j=0}^{k-1}\ \psi(j)\ \Gamma u(k-j-1) \end{aligned}$$

Solution by using z-transform

$$x(k+1) = \phi\ x(k) + \Gamma u(k)$$

Taking the z transform of both sides

$$z\ X(z) - z\ x(0) = \phi\ X(z) + \Gamma\ U(z)$$

where $X(z) = \mathcal{Z}[x(k)]\ and\ U(z) = \mathcal{Z}[u(k)]$

$\Rightarrow \qquad (z\ I - \phi)\ X(z) = z\ x(0) + \Gamma\ U(z)$

$\Rightarrow \qquad X(z) = (z\ I - \phi)^{-1} z\ x(0) + (z\ I - \phi)^{-1}\Gamma\ U(z)$

Taking $\mathcal{Z}^{-1}$ transform

$$x(k) = \mathcal{Z}^{-1}[(z\ I - \phi)^{-1} z] x(0) + \mathcal{Z}^{-1}[(z\ I - \phi)^{-1}\Gamma\ U(z)]$$

Comparing these terms with those obtained in the previous solution

$$\phi^k = \mathcal{Z}^{-1}[(z\ I - \phi)^{-1} z]$$

and

$$\sum_{j=0}^{k-1} \phi^{k-j-1}\ \Gamma\ u(j) = \mathcal{Z}^{-1}[(z\ I - \phi)^{-1}\ \Gamma\ U(z)]$$

Pulse transfer function matrix

From above, we see that if we set the initial conditions to zero, then

$$X(z) = (z\ I - \phi)^{-1}\ \Gamma\ U(z)$$

and

$$Y(z) = H\ (z\ I - \phi)^{-1}\Gamma\ U(z)$$

let

$T(z) = H(z\ I - \phi)^{-1}\Gamma$ is **The Pulse Transfer Function Matrix**

$= H\frac{adj(z\ I - \phi)}{|z\ I - \phi|}\Gamma$

So the poles of T(z) are the zeros of the *characteristic equation* $|z\ I - \phi| = 0$

$$|z\ I - \phi| = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0$$

The roots of the characteristic equation are the eigenvalues of ϕ

MIMO System Zeros

For the system defined, the above stated state and output equations, the $(n+m)$ x $(n+r)$ matrix
(n is number of states, m is number of outputs, and r is number of inputs)

$$E(z) = \begin{bmatrix} \phi - zI & \Gamma \\ H & 0 \end{bmatrix} \qquad \text{is called the system matrix}$$

The values of z that make

$$rank\ E(z) < n + min(m, r)$$

are called the zeros of the system.

If $u(k)$ and $y(k)$ are scalar $(r = 1,\ m = 1)$, then
E(z) is an $(n+1)$ x $(n+1)$ matrix. The determinant of which is

$$|E(z)| = \begin{vmatrix} \phi - zI & \Gamma \\ H & 0 \end{vmatrix}$$

$$\left(\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{cases} |A|\ |D - CA^{-1}B| & if|A| \neq 0 \\ |D|\ |A - BD^{-1}C| & if|D| \neq 0 \end{cases}\right)$$

$$\begin{aligned}
&= |\phi - zI|\ |-H(\phi - zI)^{-1}\Gamma| \\
&= (-1)^n\ |zI - \phi|\ |H(zI - \phi)^{-1}\Gamma| \qquad since\ |kA| = k^n|A| \\
&= (-1)^n\ |zI - \phi|\ \left|H\frac{adj(zI - \phi)\Gamma}{|zI - \phi|}\right| \\
&= (-1)^n\ H\ adj(zI - \phi)\Gamma
\end{aligned}$$

The values of z that make the rank of E(z) less than $n + 1$, that is the values that make $|E(z)| = 0$, are the zeros of the system.

$\Rightarrow$ the values of z that satisfy
$H\ adj(z\ I - \phi)\ \Gamma = 0$ are the system zeros.

Weighting sequence matrix

$$Y(z) = T(z)\ U(z), \qquad where \qquad T(z) = H\ (zI - \phi)^{-1}\ \Gamma$$

or

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \\ \vdots \\ Y_m(z) \end{bmatrix} = \begin{bmatrix} T_{11}(z) & T_{12}(z) & \cdots & T_{1r}(z) \\ T_{21}(z) & T_{22}(z) & \cdots & T_{2r}(z) \\ \vdots & \vdots & \vdots & \vdots \\ T_{m1}(z) & T_{m2}(z) & \cdots & T_{mr}(z) \end{bmatrix} \begin{bmatrix} U_1(z) \\ U_2(z) \\ \vdots \\ U_r(z) \end{bmatrix}$$

thus, the i-th output $Y_i(z)$ is given by

$$Y_i(z) = \sum_{j=1}^{r} T_{ij}(z)\ U_j(z) \qquad i = 1,\ 2, \cdots,\ m$$

Now

$$(zI - \phi)^{-1} = Iz^{-1} + \phi z^{-2} + \phi^2 z^{-3} + \cdots$$

$$\Rightarrow \qquad T(z) = H\Gamma z^{-1} + H\phi\Gamma z^{-2} + H\phi^2\Gamma z^{-3} + \cdots$$

The wighting sequence matrix T(k) is given by

$$T(k) = \mathcal{Z}^{-1}\{T(z)\}$$

now

$$\begin{aligned}
T(z) &= \sum_{k=0}^{\infty} T(k) z^{-k} \\
&= T(0) + T(1)z^{-1} + T(2)z^{-2} + \cdots + T(k)z^{-k} + \cdots
\end{aligned}$$

$\Rightarrow$

$$T(0) = 0$$
$$T(1) = H\Gamma$$
$$T(2) = H\phi\Gamma$$
$$\vdots$$
$$T(k) = H\phi^{k-1}\Gamma$$

$\Rightarrow$ the weighting sequence matrix is given by

$$T(k) = \begin{cases} 0 & k \leq 0 \\ H\phi^{k-1}\Gamma & k = 1,\ 2,\ 3, \cdots \end{cases}$$