Chapter 2
z-TRANSFORM

One-sided z-transform

$$\begin{aligned} X(z) &= Z[x(t)] = Z[x(kT)] = Z[x(k)] \\ &= \sum_{k=0}^{\infty} x(kT)z^{-k} = \sum_{k=0}^{\infty} x(k)z^{-k} \end{aligned}$$

Two-sided z-transform

$$X(z) = \sum_{k=-\infty}^{\infty} x(kT)z^{-k} = \sum_{k=-\infty}^{\infty} x(k)z^{-k}$$

Note that $X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots + x(kT)z^{-k} + \cdots$

Inverse z-transform

$$\begin{aligned} \mathcal{Z}^{-1}[X(z)] &= x(kT) = x(k) \\ &= \frac{1}{2\pi j} \oint_c X(z)z^{k-1}dz \end{aligned}$$

Where c is a circle with its center at the origin of the z plane such that all poles of $X(z)z^{k-1}$ are inside it

Z Transform of elementary functions:
Unit step function

$$x(t) = \begin{cases} 1(t) & 0 \le t \\ 0 & t < 0 \end{cases}$$

$$\Rightarrow X(z) = \mathcal{Z}[1(t)] = \sum_{k=0}^{\infty} 1\ z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Region of convergence $|z| > 1$

Geometric series $\quad a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \qquad |r| < 1$

Exponential Function

$$x(t) = \begin{cases} e^{-at} & 0 \le t \\ 0 & t < 0 \end{cases}$$

$$x(kT) = e^{-akT},\ k = 0, 1, 2, \cdots$$

$$\begin{aligned} X(z) = \mathcal{Z}[e^{-at}] &= \sum_{k=0}^{\infty} x(kT)z^{-k} = \sum_{k=0}^{\infty} e^{-akT}z^{-k} \\ &= \frac{1}{1 - e^{-aT}z^{-1}} \\ &= \frac{z}{z - e^{-at}} \end{aligned}$$

• See table of z-transforms on page 29 and 30 (new edition), or page 49 and 50 (old edition).

The z-transform X(z) and its inverse x(k) have a one-to-one correspondence, however, the z-transform X(z) and its inverse z-transform x(t) do not have a unique correspondence.

Properties and theorems of the z-transform

- Multiplication by a constant: $\mathcal{Z}[ax(t)] = aX(z)$
- Linearity: $\mathcal{Z}[\alpha f(k) + \beta g(k)] = \alpha F(z) + \beta G(z)$
- Multiplication by a^k : $\mathcal{Z}[a^k x(k)] = X(a^{-1}z)$
- Real translation theorem (shifting theorem):

  $If\ x(t) = 0\ for\ t < 0$

  $$\mathcal{Z}[x(t - nT)] = z^{-n}X(z)$$

  and

  $$\mathcal{Z}[x(t + nT)] = z^n\ [X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}]$$

- Initial value theorem:

  $$x(0) = \lim_{z \to \infty} X(z)$$

- Final value theorem:

  $$\lim_{k \to \infty} x(k) = \lim_{z \to 1} [(1 - z^{-1})X(z)]$$

- Real convolution Theorem:
  let

  $$x_1(t) = 0\ for\ t < 0$$

  $$x_2(t) = 0\ for\ t < 0$$

  then

  $$X_1(z)\ X_2(z) = \mathcal{Z}[\sum_{h=0}^{k} x_1(hT)\ x_2(kT - hT)]$$

INVERSE z TRANSFORM

Different Methods

1. Direct division method (Power Series Method)
2. Computational method
3. Partial-fraction-expansion method
4. Inversion integral method

• Direct division method

Express $X(z)$ in powers of z^{-1}

Example 1

$$Find\ \mathcal{Z}^{-1}\ of\ X(z) = 1 + 2z^{-1} + 3\ z^{-2} + 4\ z^{-3}$$

Solution:

$$x(0) = 1; \quad x(1) = 2; \quad x(2) = 3; \quad x(3) = 4$$

Example 2

$$Find\ \mathcal{Z}^{-1}\ of\ X(z) = \frac{10z + 5}{(z-1)(z-0.2)}$$

Solution:

$$X(z) = \frac{10\ z^{-1} + 5\ z^{-2}}{1 - 1.2\ z^{-1} + 0.2\ z^{-2}}$$

$\Rightarrow X(z) = 10z^{-1} + 17\ z^{-2} + 18.4\ z^{-3} + 18.68\ z^{-4} + \cdots$

$x(0) = 0$
$x(1) = 10$
$x(2) = 17$
$x(3) = 18.4$
$x(4) = 18.68$

• Computational method

$$X(z) = \frac{10\ z + 5}{(z-1)(z-0.2)}$$

Solution:

$$Let\ X(z) = \frac{10z + 5}{z^2 - 1.2\ z + 0.2}\ U(z)$$

where $U(z) = 1$
now, $\quad U(z) = u(0) + u(1)z^{-1} + u(2)z^{-2} + \cdots + u(k)z^{-k} + \cdots$

$\Rightarrow$ *for* $U(z) = 1$
$\Rightarrow u(0) = 1$
$\Rightarrow u(k) = 0, \qquad for\ k = 1, 2, 3 \cdots$

Converting to difference equation

$$x(k+2) - 1.2\ x(k+1) + 0.2\ x(k) = 10\ u(k+1) + 5\ u(k) \qquad (*)$$

now, $\quad$ let $k = -2$

$$\Rightarrow x(0) - 1.2\ x(-1) + 0.2\ x(-2) = 10\ u(-1) + 5\ u(-2)$$

now, $\quad x(-1) = x(-2) = 0$ and $u(-1) = u(-2) = 0$

$$\Rightarrow x(0) = 0$$

Similarly, we find

$$x(1) = 10$$

We may continue the process to find $x(k),\ k = 2,\ 3, \cdots\ using\ (*)$

• Partial Fraction Expansion

To find the $\mathcal{Z}^{-1}X(z)$, we may expand $\frac{X(z)}{z} or\ X(z)$ into partial fractions. $\frac{X(z)}{z}$ is expanded since each of the expanded terms is generally available in z-transform tables.

Alternatively, X(z) may be expanded and use of the shifting theorem may be made.

Example

$$X(z) = \frac{z^{-1}}{1 - az^{-1}}$$

let $\quad Y(z) = z\ X(z) = \frac{1}{1-az^{-1}}$

$$\Rightarrow \mathcal{Z}^{-1}\{Y(z)\} = y(k) = a^k$$

now,

$$X(z) = z^{-1}\ Y(z)$$

$$\Rightarrow \mathcal{Z}^{-1}\{X(z)\} = x(k) = y(k-1) = a^{k-1}$$

thus,

$$x(k) = \begin{cases} a^{k-1} & k = 1,\ 2,\ 3, \cdots \\ 0 & k \leq 0 \end{cases}$$

General procedure for partial fraction expansion:

Given $X(z)$, $find\ \frac{X(z)}{z}$
let

$$\frac{X(z)}{z} = \frac{a_0 + a_1 z + \cdots + a_N z^N}{b_0 + b_1 z + \cdots + b_M z^M} \tag{1}$$

If $M > N$, no adjustment need be made to $\frac{X(z)}{z}$,
If $N > M$, we divide through

$$\frac{X(z)}{z} = c_{N-M}\ z^{N-M} + c_{N-M-1}\ z^{N-M-1} + \cdots + c_1\ z + c_0$$
$$\underbrace{+\frac{d_0 + d_1 z + \cdots + d_{M-1} z^{M-1}}{b_0 + b_1 z + \cdots + b_M z^M}}_{=\psi(z)}$$

Factoring $\psi(z)$ where we have one repeated pole of order k, call it z_r, and the rest unique, $z_{k+1},\ z_{k+2},\ \cdots, z_m$

$$\psi(z) = \frac{A_{1k}}{(z-z_r)^k} + \frac{A_{1k-1}}{(z-z_r)^{k-1}} + \cdots + \frac{A_{11}}{z-z_r} + \sum_{j=k+1}^{M} \frac{A_j}{z-z_j} \tag{2}$$

Where $\quad A_{1j} = \frac{1}{(k-j)!}\ [\frac{d^{k-j}}{dz^{k-j}}\ (z-z_j)^k\ \psi(z)]\ |_{z=z_j}, \qquad j = 1,\ 2,\ \cdots,\ k$

$$A_j = (z-z_j)\ \psi(z)\ |_{z=z_j}, \qquad j = k+1,\ k+2,\ \cdots,\ M$$

Substituting (3) into (2) and multiplying by z and taking inverse transform gives us:

$$\begin{aligned}
\mathcal{Z}^{-1}[X(z)] &= x(n) \\
&= \mathcal{Z}^{-1}\left[c_{N-M}\ z^{N-M+1} + c_{N-M-1}\ z^{N-M} + \cdots + c_1\ z^2 + c_0\ z\right] \\
&+ \mathcal{Z}^{-1}\left[\sum_{j=1}^{k} \frac{A_{1j}\ z}{(z-z_r)^j}\right] + \mathcal{Z}^{-1}\left[\sum_{j=k+1}^{M} \frac{A_j\ z}{z-z_r}\right]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow x(n) &= \sum_{n=M}^{N} C_{N-M}\ \delta(n+(N-M+1)) \\
&+ [A_{11}\ z_r^N + A_{12}\ n\ z_r^{n-1} + \cdots + \frac{A_{1k}\ n(n-1)\cdots(n-(k-2))\ z_r^{n-k+1}}{(k-1)!} \\
&+ \sum_{j=k+1}^{M} A_j\ z_j^n\ ]\ u(n)
\end{aligned}$$

Where the following has been used

$$\mathcal{Z}^{-1}\left\{\frac{z}{(z-a)^k}\right\} = \frac{n\ (n-1)\cdots\ (n-(k-2))\ a^{n-k+1}\ u(n)}{(k-1)!}$$

where $u(n)$ is the unit step function.

Example

Find $\mathcal{Z}^{-1}\{X(z)\}$ where,

$$X(z) = \frac{z^4 + z^2}{(z - \frac{1}{2})(z - \frac{1}{4})}$$

Solution

$$\frac{X(z)}{z} = \frac{z^3 + z}{z^2 - \frac{3}{4}z + \frac{1}{8}} = z + \frac{3}{4} + \frac{\frac{23}{16}z - \frac{3}{32}}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

now,

$$\frac{\frac{23}{16}z - \frac{3}{32}}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{A_1}{z - \frac{1}{2}} + \frac{A_2}{z - \frac{1}{4}}$$

Where

$$A_1 = \frac{\frac{23}{16}\ z - \frac{3}{32}}{z - \frac{1}{4}}\ |_{z=\frac{1}{2}} = \frac{\frac{5}{8}}{\frac{1}{4}} = \frac{5}{2}$$

$$A_2 = \frac{\frac{23}{16}\ z - \frac{3}{32}}{z - \frac{1}{2}}\ |_{z=\frac{1}{4}} = \frac{\frac{17}{64}}{-\frac{1}{4}} = -\frac{17}{16}$$

Thus

$$\frac{X(z)}{z} = z + \frac{3}{4} + \frac{\frac{5}{2}}{z - \frac{1}{2}} - \frac{\frac{17}{16}}{z - \frac{1}{4}}$$

$$\begin{aligned} x(n) &= \mathcal{Z}^{-1}\left[z^2 + \frac{3}{4}z\right] + \mathcal{Z}^{-1}\left[\frac{\frac{5}{2}z}{z - \frac{1}{2}} - \frac{\frac{17}{16}z}{z - \frac{1}{4}}\right] \\ &= \delta(n+2) + \frac{3}{4}\ \delta(n+1) + \left[\frac{5}{2}\ (\frac{1}{2})^n - \frac{17}{16}\ (\frac{1}{4})^n\right]\ u(n) \end{aligned}$$

• Inversion integral method

Background material:

Suppose z_0 is an isolated singular point (pole) of F(z). Expand F(z) in a Laurent series about $z = z_0$

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

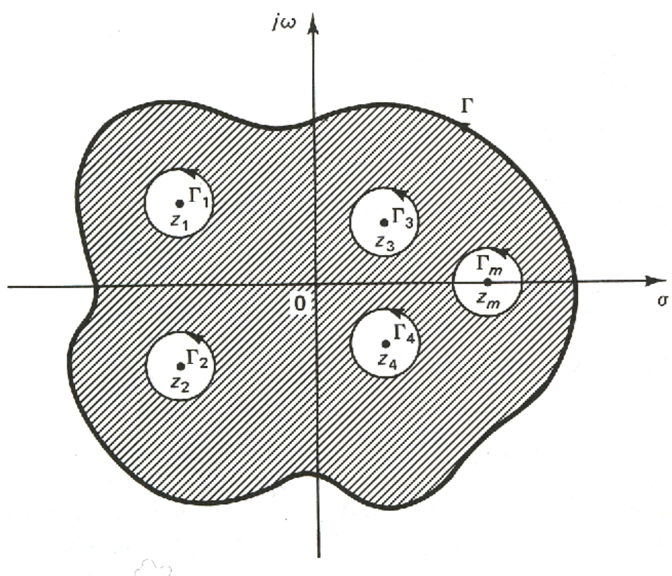
$$a_n = \frac{1}{2\pi j} \oint_{\Gamma_1} \frac{F(z)}{(z - z_0)^{n+1}}\, dz \qquad n = 0,\ 1,\ 2, \cdots$$

$$b_n = \frac{1}{2\pi j} \oint_{\Gamma_2} \frac{F(z)}{(z - z_0)^{-n+1}}\, dz \qquad n = 1,\ 2,\ 3, \cdots$$

where Γ_1 and Γ_2 are closed paths around z_0
and

$$b_1 = \frac{1}{2\pi j} \oint_{\Gamma} F(z)\ d(z)$$

where Γ is any closed path within and on which F(z) is analytic except at $z = z_0$, *and* b_1 is called the *residue* of F(z) at the pole z_0.



Now

$$\begin{aligned} \oint_{\Gamma} F(z)\ dz &= \oint_{\Gamma_1} F(z)\ dz + \oint_{\Gamma_2} F(z)\ dz + \ \cdots + \oint_{\Gamma_m} F(z)\ dz \\ &= 2\pi j (b_{1_1} + b_{1_2} + \cdots + b_{1_m}) \qquad \Leftarrow \textit{Residue theorem} \end{aligned}$$

• Inversion integral

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k} = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ \cdots + x(kT)\ z^{-k} + \cdots$$

$\Rightarrow$

$$X(z)z^{k-1} = x(0)z^{k-1} + x(T)z^{k-2} + x(2T)z^{k-3} + \cdots + x(kT)z^{-1} + \cdots$$

Note, this $\nearrow$ is the Laurent series expression of $X(z)z^{k-1}$ around point $z = 0$, and x(kT) is the residue

$\Rightarrow \qquad x(kT) = \frac{1}{2\pi j} \oint_c X(z)z^{k-1}\ dz$

the inverse $\nearrow$ integral for the z-transform

Inverse z transform using inversion integral

$$x(k) = x(kT) = \sum_{i=1}^{M} [residue\ of\ X(z)z^{k-1}\ at\ pole\ z = z_i\ of\ X(z)z^{k-1}\ ]$$

assuming M poles.

The residue K, for simple pole is given by

$$K = \lim_{z \to\ z_i} [(z - z_i)X(z)z^{k-1}]$$

The residue K, for multiple pole z_j of order q is given by

$$K = \frac{1}{(q-1)!}\ \lim_{z \to\ z_j} \frac{d^{q-1}}{dz^{q-1}}\ [(z - z_j)^q X(z)z^{k-1}]$$

Example

Find $\mathcal{Z}^{-1}[X(z)],\ where\ X(z) = \frac{z^2}{(z-1)^2(z-e^{-aT})}$

Solution:

$$X(z)z^{k-1} = \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})}$$

Simple pole at $z \ = \ e^{-aT}$
Double pole at $z \ = \ 1$

$$\begin{aligned} x(k) &= \sum_{i=1}^{2} \left[residue\ of \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})}\ at\ pole\ z = z_i \right] \\ &= K_1 + K_2 \end{aligned}$$

where

$$\begin{aligned} K_1 &= \lim_{z \to e^{-aT}} \left[(z-e^{-aT})\ \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})} \right] = \frac{e^{-a(k+1)T}}{(1-e^{-aT})^2} \\ K_2 &= \frac{1}{(2-1)!}\ \lim_{z \to 1}\ \frac{d}{dz}\ \left[(z-1)^2 \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})} \right] \\ &= \frac{k}{1-e^{-aT}} - \frac{e^{-aT}}{(1-e^{-aT})^2} \longrightarrow see\ steps\ below \end{aligned}$$

$$\Rightarrow x(kT) = \frac{kT}{T(1-e^{-aT})} - e^{-aT}\frac{(1-e^{-akT})}{(1-e^{-aT})^2} \qquad k = 0,\ 1,\ 2,\ \cdots$$

- Steps

$$d\frac{v}{u} = \frac{udv - vdu}{u^2}$$

$$\begin{aligned} & \lim_{z \to 1} \frac{d}{dz} \left(\frac{z^{k+1}}{z-e^{-aT}} \right) \\ = & \lim_{z \to 1} \frac{(k+1)z^k(z-e^{-aT}) - z^{k+1}}{(z-e^{-aT})^2} \\ = & \lim_{z \to 1} \left[\frac{(k+1)z^k}{z-e^{-aT}} - \frac{z^{k+1}}{(z-e^{-aT})^2} \right] \\ = & \frac{k+1}{1-e^{-aT}} - \frac{1}{(1-e^{-aT})^2} \\ = & \frac{k}{1-e^{-aT}} + \frac{1-e^{-aT}}{(1-e^{-aT})^2} - \frac{1}{(1-e^{-aT})^2} \\ = & \frac{k}{1-e^{-aT}} - \frac{e^{-aT}}{(1-e^{-aT})^2} \end{aligned}$$

• **<u>Pulse-Transfer Function</u>**

Difference equation:

$$\begin{aligned}&x(k) + a_1 x(k-1) + \cdots + a_n x(k-n)\\ &= b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)\end{aligned}$$

Taking z transform

$$X(z) + a_1 z^{-1} X(z) + \cdots + a_n z^{-n} X(z)$$

$$= b_0 U(z) + b_1 z^{-1} U(z) + \cdots + b_n z^{-n} U(z)$$

$$G(z) \equiv \frac{X(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

$\nwarrow$ *Pulse Transfer Function*

Now, Kronecker delta function $\delta_0(kT)$

$$\delta_0(kT) = \begin{cases} 1 & \textit{for } k = 0 \\ 0 & \textit{for } k \neq 0 \end{cases}$$

$$\mathcal{Z}\left[\delta_0(kT)\right] = 1$$

$\Rightarrow$ G(z) is the z transform of the response to $\delta_0(kT)$. It is called the **pulse transfer function**

$g(k) = \mathcal{Z}^{-1}\{G(z)\}$ is called the **weighting sequence**.

z transform method of solving difference equations

Example

Solve: $x(k+2) + 3x(k+1) + 2x(k) = 0; \qquad x(0) = 0,\ x(1) = 1$

Solution

taking the z transform

$$z^2\ X(z) - z^2\ x(0) - z\ x(1) + 3z\ X(z) - 3z\ x(0) + 2X(z) = 0$$

Substituting initial data

$$X(z) = \frac{z}{z^2 + 3z + z} = \frac{z}{(z+1)(z+2)} = \frac{z}{z+1} - \frac{z}{z+2}$$
$$= \frac{1}{1 + z^{-1}} - \frac{1}{1 + 2z^{-1}}$$

$$\mathcal{Z}^{-1}\left[\frac{1}{1+z^{-1}}\right] = (-1)^k, \qquad \mathcal{Z}^{-1}\left[\frac{1}{1+2z^{-1}}\right] = (-2)^k$$

$$\Rightarrow \qquad x(k) = (-1)^k - (-2)^k, \qquad k = 0,\ 1,\ 2,\ \cdots$$