ECE452/552

Review

Topics

1. **z Transform**

   - Properties

2. **Inverse z Transform by:**

   (a) Direct division method
   (b) Computational method
   (c) Partial-fraction expansion
   (d) Inversion integral

3. **Given difference equation, find:**

   (a) Pulse-transfer function (weighting sequence)
   (b) Solution

4. (a) Impulse sampling
   (b) Data hold

$$\left.\begin{array}{l}\bullet\ ZOH\\ \bullet\ FOH\end{array}\right\} Transfer functions.$$

   (c) Data reconstruction
   - Sampling theorem
   - Aliasing

5. (a) Convolution summation
   (b) Starred Laplace transform and pulse transfer function

$$X^{\star}(s) = X(z)$$

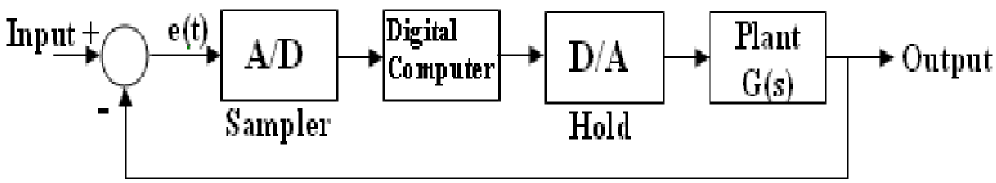
   (c) Methods for obtaining the z transform
   i. Definition
   ii. Using partial fractions
   iii. Using residues

6. **Block diagram reduction of sampled data system**

- Feedback systems

7. **Obtaining response between consecutive sampling instants**
   (a) Laplace transform
   (b) Modified Z transform
8. **Stability tests**
   (a) Jury test
   (b) Bilinear transformation and Routh criterion

Typical Model



1. **The z Transform**

$$\begin{aligned} X(z) &= \mathcal{Z}[x(t)] = \mathcal{Z}[x(kT)] = \mathcal{Z}[x(k)] \\ &= \sum_{k=0}^{\infty} x(kT)z^{-k} = \sum_{k=0}^{\infty} x(k)z^{-k} \end{aligned}$$

Geometric Series:

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r} \qquad |r| < 1$$

2. **Inverse z Transform**

$$\mathcal{Z}^{-1}[X(z)] = x(kT) = x(k)$$

(a) Inversion integral

$$\begin{aligned} x(k) &= \frac{1}{2\pi j} \oint_c X(z) z^{k-1} dz \\ &= \sum_{i=1}^{M} [residue\ of\ X(z)\ z^{k-1}\ at\ pole\ z = z_i\ of\ X(z)z^{k-1}] \\ & \quad assuming\ M\ poles \end{aligned}$$

Residues, K

i. For simple pole

$$K = \lim_{z \to z_i} [(z - z_i)X(z)z^{k-1}]$$

ii. For multiple pole z_j of order q

$$K = \frac{1}{(q-1)!} \lim_{z \to z_j} \frac{d^{q-1}}{dz^{q-1}} [(z - z_j)^q\ X(z)z^{k-1}]$$

(b) Direct division

Note

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty} x(k) z^{-k} \\ &= x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ldots + x(kT)z^{-k} + \ldots \end{aligned}$$

(c) Computational method

Express as a difference equation and use it along with initial conditions to give response.

(d) Partial fractions

Given G(z), express $\frac{G(z)}{z}$ as a sum of simpler terms for which the inverse transform is available in tables.

3. (a) Pulse transfer function

$$G(z) = \frac{X(z)}{U(z)} \;\frac{\leftarrow \textit{Output}}{\leftarrow \textit{Input}}$$

If $u(kT) = \delta_o(kT) \leftarrow \textit{Kronecker delta}$

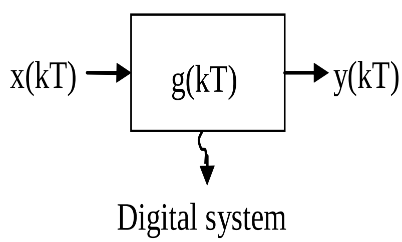
$\Rightarrow U(z) = 1$

$\Rightarrow G(z) = \mathcal{Z}[\text{unit impulse response}]$

Also

$g(k) = \mathcal{Z}^{-1}\{G(z)\} \leftarrow$ weighting sequence

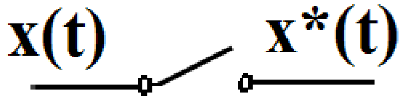
Convolution Summation



$$\begin{aligned} y(kT) &= \sum_{h=0}^{\infty} g(kT - hT)\; x(hT) \\ &= \sum_{h=0}^{\infty} x(kT - hT)\; g(hT) \\ &= x(kT) * g(kT) \end{aligned}$$

(b) **Solving difference equations**

Take Z transform, partial fraction expansion, sum up $\mathcal{Z}^{-1}$ of terms



4. (a) **Impulse Sampling**

$$
\begin{aligned}
x^*(t) &= \sum_{k=-\infty}^{\infty} x(t)\ \underbrace{\delta(t-kT)}_{\text{Dirac delta}} \\
&= x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)
\end{aligned}
$$

Or

$$x^*(t) = \sum_{k=-\infty}^{\infty} x(kT)\ \delta(t-kT)$$

$$X^*(s) = \mathcal{L}[x^*(t)] = \sum_{k=0}^{\infty} x(kT)\ e^{-kTs}$$

$$c.f.\ z\ transform \Rightarrow if\ e^{Ts}\ =\ z$$

then

$$X^*(s)\ |_{s=\frac{1}{T}\ lnz} = X(z)$$

(b) **Data hold**

$$\left.\begin{array}{l} \bullet\ ZOH \\ \bullet\ FOH \end{array}\right\} Transfer\ functions.$$

(c) **Data reconstruction**

$$X^*(jw) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(jw + jw_s k)$$

$where\ w_s = \frac{2\pi}{T}$

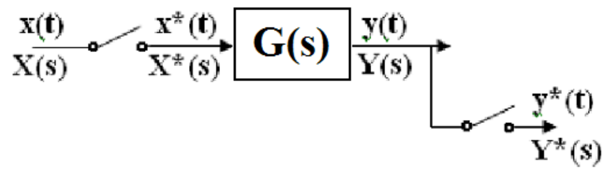
- Aliasing
- Low-Pass Filtering
- ZOH as a LPF

5. (a) **Convolution Summation**

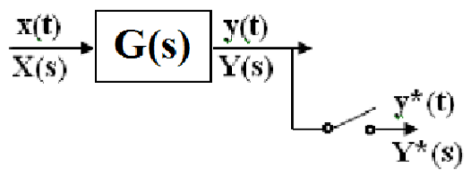
(b) **Starred Laplace transform**

$$X^*(s) = \mathcal{L}(x^*(t))$$

Y(s) = G(s) X*(s)



$$\begin{aligned} Y^*(s) &= [G(s)\ X^*(s)]^* \\ &= G^*(s)\ X^*(s) \\ &= G(z)\ X(z) \end{aligned}$$



$$Y(s) = G(s)\ X(s)$$

$$Y^*(s) = [G(s)X(s)]^* = [G\ X(s)]^*\ =\ G\ X(z)$$

(c) **Methods of obtaining z transform**

i. definition

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k}$$

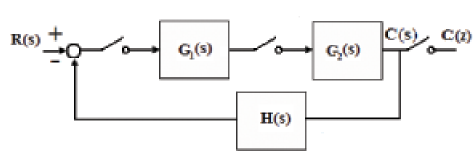
ii. partial fractions

$$\begin{aligned} X(z) &= \mathcal{Z}[X(s)\ expanded\ into\ partial\ fractions\ X_i(s)] \\ &= \sum_i \mathcal{Z}[X_i(s)]\ \rightarrow use\ tables \end{aligned}$$

iii. residues

$$X(s) = \sum [residues\ of\ \frac{X(s)\ z}{z - e^{Ts}}\ at\ pole\ of\ X(s)]$$

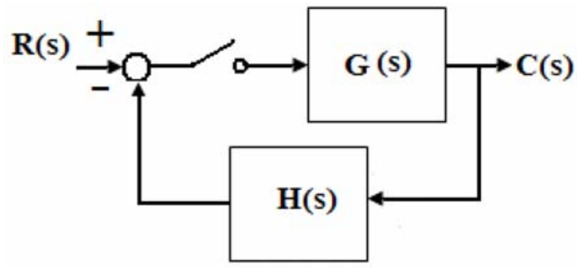
6. **Block diagram reduction**



$$C(z) = \frac{G_1(z)\ G_2(z)\ R(z)}{1 + G_1(z)\ G_2H(z)}$$

7. **Response between sampling instants**

- Laplace transform
  e.g.



$$c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}[G(s)\frac{R^*(s)}{1+GH^*(s)}]$$

- Modified z transform
  (a) $G(z,m) = \mathcal{Z}^{-1}\sum\left[residue\ of\ \frac{G(s)e^{mTs}z}{z-e^{Ts}}\ at\ pole\ of\ G(s)\right]$

$$G(z) = \lim_{m\to 0} z\, G(z,m)$$

  (b) Inverse transform using division

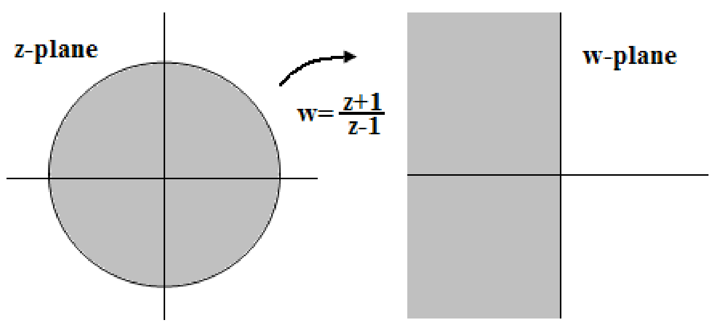
$$Y(z,m) = y_0(m)z^{-1} + y_1(m)z^{-2} + y_2(m)z^{-3} + \ldots$$

8. **Stability**

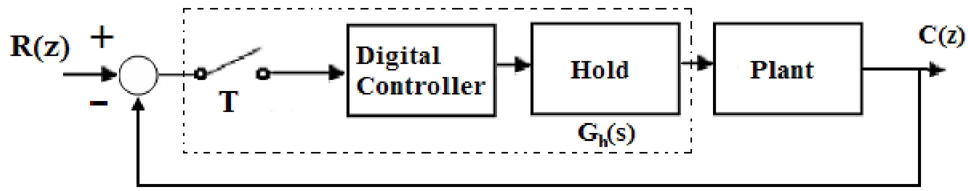
   (a) Jury test
   - be able to set up table and read results

   (b) Bilinear transformation and Routh criterion

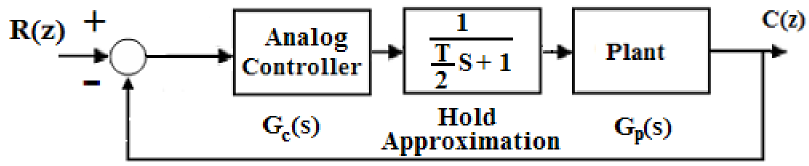
$$z = \frac{w+1}{w-1}$$



Summary: Design of discrete time control systems via transform methods



Obtaining discrete time equivalents of continuous time controllers



continuous-time control system modified to allow for time lag of hold

Design Procedure:

1. Design analog controller for the above system
2. Digitize the controller using one of s to z transformations
3. Perform computer simulation of system to check performance
4. If performance is not adequate, use a different s-to-z mapping
5. Iterate steps (3) and (4) until adequate performance is achieved

TABLE 4–1 EQUIVALENT DISCRETE-TIME FILTERS FOR A CONTINUOUS-TIME FILTER $G(s) = a/(s + a)$

Mapping method	Mapping equation	Equivalent discrete-time filter for $G(s) = \frac{a}{s+a}$
Backward difference method	$s = \frac{1-z^{-1}}{T}$	$G_D(z) = \frac{a}{\frac{1-z^{-1}}{T} + a}$
Forward difference method	$s = \frac{1-z^{-1}}{Tz^{-1}}$	This method is not recommended, because the discrete-time equivalent may become unstable.
Bilinear transformation method	$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$	$G_D(z) = \frac{a}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + a}$
Bilinear transformation method with frequency prewarping	$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ $\left(\omega_A = \frac{2}{T}\tan\frac{\omega_D T}{2}\right)$	$G_D(z) = \frac{\tan\frac{aT}{2}}{\frac{1-z^{-1}}{1+z^{-1}} + \tan\frac{aT}{2}}$
Impulse-invariance method	$G_D(z) = T\,\mathcal{Z}[G(s)]$	$G_D(z) = \frac{Ta}{1-e^{-aT}z^{-1}}$
Step-invariance method	$G_D(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}G(s)\right]$	$G_D(z) = \frac{(1-e^{-aT})z^{-1}}{1-e^{-aT}z^{-1}}$
Matched pole-zero mapping method	A pole or zero at $s = -a$ is mapped to $z = e^{-aT}$. An infinite pole or zero is mapped to $z = -1$.	$G_D(z) = \frac{1-e^{-aT}}{2}\frac{1+z^{-1}}{1-e^{-aT}z^{-1}}$

Design based on the frequency response method

- Bilinear transformation and the w-plane

$$let\ z = \frac{1 + \frac{T}{2}\mathrm{w}}{1 - \frac{T}{2}\mathrm{w}},\ T\ is\ the\ sampling\ period$$

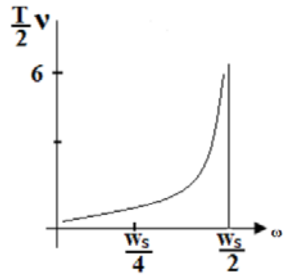
The inverse transformation is

$$\mathrm{w} = \frac{2}{T}\frac{z-1}{z+1}$$

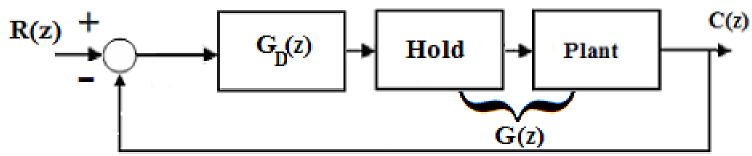
The w plane resembles the s plane geometrically, however the frequency axis in the w plane is distorted.

$$\begin{aligned}\mathrm{w}\,|_{\mathrm{w}=j\nu} &= j\nu = \frac{2}{T}\,\frac{z-1}{z+1}\,|_{z=e^{j\omega T}} = \frac{2}{T}\,\frac{e^{j\omega T}-1}{e^{j\omega T}+1}\\ &= \frac{2}{T}\,\frac{e^{j\frac{\omega T}{2}} - e^{-j\frac{\omega T}{2}}}{e^{j\frac{\omega T}{2}} + e^{-j\frac{\omega T}{2}}} = \frac{2}{T}j\tan\frac{\omega T}{2}\end{aligned}$$

ν is a fictitious frequency



Design procedure in the w-plane



1. Obtain G(z), the z transform of the plant preceded by a hold. Then transform G(z) into a transfer function G(w)

$$G(\mathrm{w}) = G(z) \mid_{z=\frac{1+\frac{T}{2}\mathrm{w}}{1-\frac{T}{2}\mathrm{w}}}$$

   Choose a T about 10 times the bandwidth of the closed loop system.

2. Substitute $\mathrm{w} = j\nu$ $into$ $G(\mathrm{w})$ and $plot$ the $Bode$ $diagram$ for $G(j\nu)$.

3. Read from the plot the gain and phase margins and the low frequency gain (which will determine static accuracy).

4. Design $G_D(\mathrm{w})$ to achieve desired loop transfer function.

5. Transform the $G_D(\mathrm{w})$ $into$ $G_D(z)$.

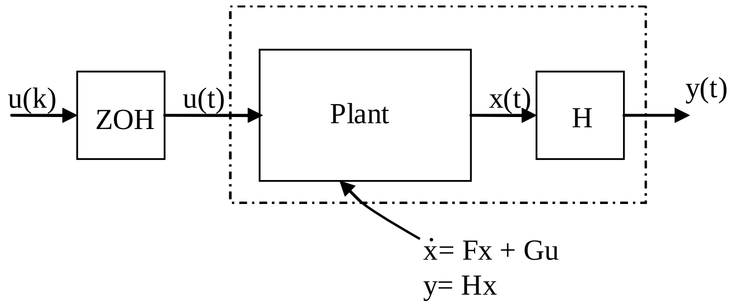
$$G_D(z) = G_D(\mathrm{w}) \mid_{\mathrm{W}=\frac{2}{T}\ \frac{z-1}{z+1}}$$

6. Realize $G_D(z)$ by a computational algorithm.

STATE-SPACE SUMMARY

For ZOH

$$u(t) = u(k); \qquad kT < t < (k+1)T$$



Difference equation

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(t) = H\ x(k)$$

$$\Phi = e^{FT} = I + FT + \frac{F^2T^2}{2!} + FT + \frac{F^3T^3}{3!} + \cdots$$

$$\Gamma = \int_0^T e^{F\eta}\ d\eta\ G$$

Solution of the state equation

$$let \quad \psi(k) = \Phi(k) \qquad \leftarrow\ the\ state\ transition\ matrix$$

$$x(k) = \Psi(k)\ x(0) + \sum_{j=0}^{k-1} \Psi(k-j-1)\ \Gamma u(j)$$

$$y(k) = H\ \Psi(k)\ x(0) + H \sum_{j=0}^{k-1} \Psi(k-j-1)\ \Gamma u(j)$$

Pulse Transfer Function

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$\Rightarrow \qquad X(z) = (zI - \Phi)^{-1}\ \Gamma U(z)$$

$$y(k) = H\ x(k)$$

$$\Rightarrow \qquad Y(z) = H\ (zI - \Phi)^{-1}\ \Gamma U(z)$$

$\Rightarrow$

pulse transfer function matrix T(z) is

$$\begin{aligned} T(z) &= H\ (zI - \Phi)^{-1}\ \Gamma \\ &= H\ \frac{adj(zI - \Phi)}{|zI - \Phi|}\Gamma \end{aligned}$$

Weighting sequence matrix

$$\begin{aligned} T(k) &= \mathcal{Z}^{-1}\{T(z)\} \\ &= \mathcal{Z}^{-1}\{H\ \Gamma z^{-1} + H\ \Phi\ \Gamma z^{-2} + H\ \Phi^2 \Gamma z^{-3} + \cdots\} \end{aligned}$$

$$\Rightarrow \qquad T(k) = \begin{cases} 0, & k \leq 0 \\ H\Phi^{k-1}\ \Gamma, & k = 1,\ 2,\ 3 \end{cases}$$

recall $\quad T(z) = \sum_{k=0}^{\infty} T(k)\ z^{-k}$

State-Space Design Summary

Design: Two steps:

1. Control law design assuming full state feedback
2. Estimator or observer design (considered full state estimator design)

Control Law:

$$u = -K\ x$$

state equation $\quad x(k+1) = \Phi\ x(k) + \Gamma\ u(k)$
$\Rightarrow$ For closed loop

$$x(k+1) = (\Phi - \Gamma\ K)x(k)$$

Poles are given by the eigenvalues of $(\Phi - \Gamma\ K)$
i.e. characteristic equation is $det(zI - \Phi + \Gamma\ K) = 0$

Pole placement

1. Matching coefficients of
$$det(zI - \Phi + \Gamma K)$$
with the desired characteristic equation
$$\alpha_c(z) = (z - \beta_1)(z - \beta_2)(z - \beta_3)\cdots$$
where $\beta_1,\ \beta_2,\ \cdots$ are pole locations.
2. Use control canonical form to ease computations
3. Ackerman's formula
$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}^{-1} \alpha_c(\Phi)$$
where $\mathcal{C} = \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots \end{bmatrix}$ is the controllability matrix,
n is the order of the system and $\alpha_c(z)$ is the desired characteristic equation.

__**Controllability**__: $\mathcal{C}$ must be rank n.

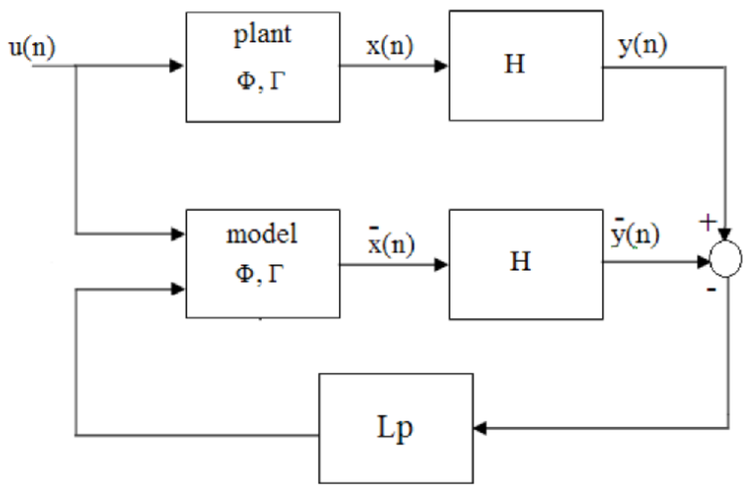
where u is scalar, $\mathcal{C}$ is an n x n matrix and if its determinant is nonzero, then the rank of $\mathcal{C}$ is n.
For multi-input system, $\mathcal{C}\mathcal{C}^T$ will give an n x n matrix, and if its determinant is non-zero, then the rank of $\mathcal{C}$ is n. $\mathcal{C} = n$ x nm, where n is # of states, and nm is # of inputs

FULL STATE ESTIMATOR DESIGN

Two kinds:

1. <u>prediction</u> estimator, $\bar{x}(k)$ is based on measurements up to $y(k-1)$
2. <u>Current</u> estimator, $\hat{x}(k)$ is based on measurements up to $y(k)$

FULL STATE ESTIMATOR DESIGN



Closed-loop Estimator

$$\bar{x}(k+1) = \Phi\bar{x}(k) + \Gamma\ u(k) + L_p[y(k) - H\ \bar{x}(k)]$$

error: $\equiv \bar{x} - x$
error estimate: $\tilde{x}(k+1) = [\Phi - L_p\ H]\ \tilde{x}(k)$

The dynamics of the error is dependent on the poles of the closed loop estimator and are given by the eigenvalues of $(\Phi - L_p\ H)$ which satisfy the characteristic equation

$$det(zI - \Phi + \ L_p\ H) = 0$$

Selection of L_p

Ackerman's formula

$$L_p = \alpha_e(\Phi) \begin{bmatrix} H \\ H\ \Phi \\ H\ \Phi^2 \\ \vdots \\ H\ \Phi^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Observability

$$O = \begin{bmatrix} H \\ H\ \Phi \\ H\ \Phi^2 \\ \vdots \\ H\ \Phi^{n-1} \end{bmatrix} \qquad must\ be\ of\ rank\ n.$$

Current Estimator

$$\hat{x}(k) = \bar{x}(k) + L_c(y(k) - H\ \bar{x}(k))$$

where

$$\bar{x}(k) = \Phi\ \hat{x}(k-1) + \Gamma\ u(k-1)$$

$\bar{x}(k)$ is the predicted estimate based on a model prediction from the previous time estimate

$$\Rightarrow \qquad \bar{x}(k+1) = \Phi\ \bar{x}(k) + \Gamma\ u(k) + \Phi\ L_c[y(k) - H\ \bar{x}(k)]$$

compare that result with the prediction estimator.

The estimation-error equation for $\bar{x}(k)$ is

$$\tilde{x}(k+1) = [\Phi - \Phi\ L_c H]\tilde{x}(k) \qquad where\ \tilde{x} = \bar{x} - x.$$

$$L_p = \Phi\ L_c$$

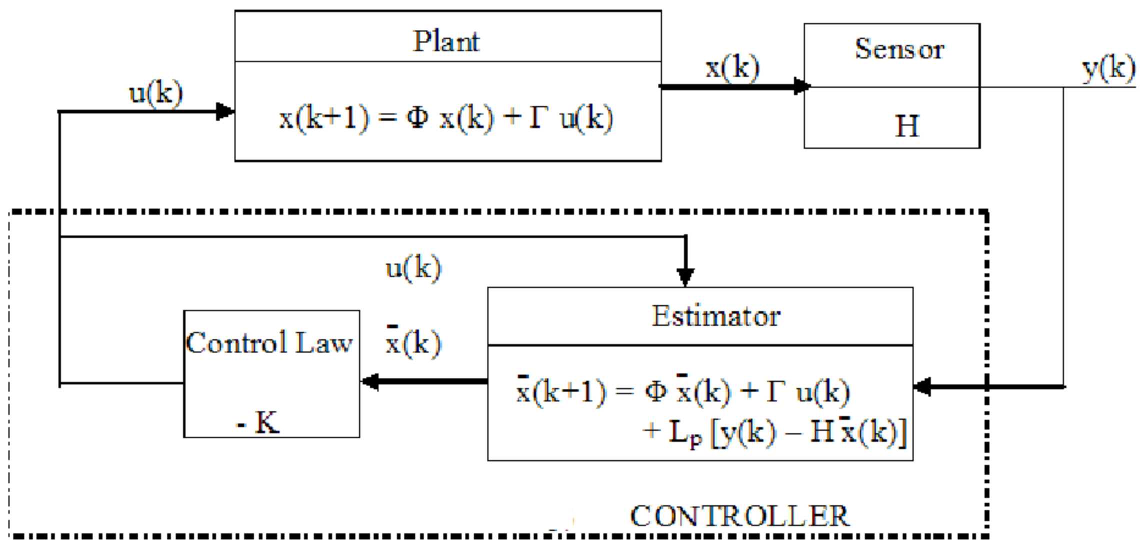
The estimation-error equation for $\hat{x}(k)$ is

$$\tilde{x}(k+1) = [\Phi - \ L_c H\ \Phi]\tilde{x}(k) \qquad where\ \tilde{x} = \hat{x} - x.$$

Using Ackerman's formula

$$L_c = \alpha_e(\Phi) \begin{bmatrix} H\Phi \\ H\Phi^2 \\ H\Phi^3 \\ \vdots \\ H\phi^n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{1}$$

Combined Control Law and Estimator



$$\begin{bmatrix} \tilde{x}(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} \Phi - L_p\, H & 0 \\ -\Gamma\, K & \Phi - \Gamma\, K \end{bmatrix} \begin{bmatrix} \tilde{x}(k) \\ x(k) \end{bmatrix}$$

- Separation principle
- Controller transfer function