

EE452/552

Project #1

The effect of sampling on transfer functions

PURPOSE: The purpose of this project is to examine the effects of sampling on the transfer function and consequently on the frequency response of a system.

Overview:

Consider a system described by a state space representation:

$$\dot{x} = Ax + Bu \quad (1)$$

$$y = Cx \quad (2)$$

Application of the Laplace transform results in the transfer function:

$$H(s) = \frac{Y(s)}{U(s)} \quad (3)$$

$$= C(sI - A)^{-1}B \quad (4)$$

We will also consider the case of using the Laplace transform of the sampled output as shown in Figure 1. The sampling period is given by T :

$$H_s(s) = \frac{Y^*(s)}{U(s)} = \frac{Y(z)|_{z=e^{sT}}}{U(s)} \quad (5)$$

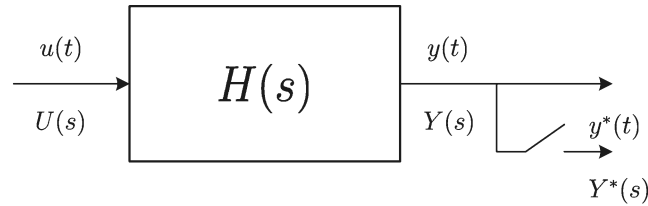


Figure 1: Continuous time system with unsampled, $y(t)$, and sampled, $y^*(t)$, outputs

We will examine the effect of using different inputs in forming the transfer functions $H(s)$ and $H_s(s)$.

Given a state space description, as shown above, the output is given by:

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau \quad (6)$$

Transfer Function, $H(s)$:

To determine a transfer function, $H(s)$, we set initial conditions to zero, so that

$$x(t_0) = 0$$

Furthermore we arbitrarily set $t_0 = 0$. This results in:

$$y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau \quad (7)$$

In the following, we consider 3 different outputs: 1) unit step, 2) sine wave, and 3) impulse function. Substituting these we obtain the output, $y(t)$, for the relevant input. Taking the Laplace transform of this and forming the quotient by dividing by the Laplace of the input results in $H(s)$.

Transfer Function, $H_s(s)$:

To determine $H_s(s)$ we start again with (6) to first find the difference equation. We set $t_0 = kT$ and $t = (k + 1)T$, where T represents the sampling period. Next, the z-transform is taken of this difference equation after the specific input is substituted. This results in finding $Y(z)$ ($= Y^*(s)$). Finally $H_s(s)$ is found by dividing by $U(s)$.

Frequency response:

Finally we will compare the magnitude frequency responses of $H_s(s) = \frac{Y^*(s)}{U(s)} = \frac{Y(z)}{U(s)} \Big|_{z=e^{j\omega T}}$ for the various inputs with that for $H(s)|_{s=j\omega}$ for a simple first order system.

Tasks:

Transfer Function, $H(s)$:

1. Unit step input, $1(t)$: Using $u(t) = 1(t)$ in (7) solve for $y(t)$. Apply the Laplace transform to this solution and subsequently form the quotient $H(s)$ as given by (3).
2. Sine wave input, $\cos(\omega_o t)$: Using $u(t) = \cos(\omega_o t)$ in (7) solve for $y(t)$. Apply the Laplace transform to this solution and subsequently form the quotient $H(s)$ as given by (3).
3. Impulse input, $\delta(t)$ (Dirac delta): Using $u(t) = \delta(t)$ in (7) solve for $y(t)$. Apply the Laplace transform to this solution and subsequently form the quotient $H(s)$ as given by (3).

Transfer Function, $H_s(s)$:

First find the general difference equation by setting $t_0 = kT$ and $t = (k + 1)T$, where T represents the sampling period, in (6).

4. Unit step input, $1(kT)$: Substitute $u(kT) = 1(kT)$ into the general difference equation. Apply the z-transform to this and subsequently form the quotient $H_s(s)$ as given by (5).
5. Sine wave input, $\cos(\omega_o kT)$: Substitute $u(kT) = \cos(\omega_o kT)$ into the general difference equation. Apply the z-transform to this and subsequently form the quotient $H_s(s)$ as given by (5).
6. Impulse input, $\delta_0(kT)$ (Kronecker delta): Substitute $u(kT) = \delta_0(kT)$ into the general difference equation. Apply the z-transform to this and subsequently form the quotient $H_s(s)$ as given by (5).

Frequency response, $H_s(s)|_{z=e^{j\omega T}}$ vs. $H(s)|_{s=j\omega}$:

We will consider the simple first order system given by the low-pass filter shown in Figure 2. Use $R = 1 \text{ k}\Omega$ and $C = 6.25 \text{ }\mu\text{F}$. The input $u(t) = v_g(t)$, the output $y(t) = v_o(t)$ and the state is $x(t) = v_o(t)$.

- Derive the system description in state space form as given by (1) and (2). That is, determine the parameters $\{A, B, C\}$.

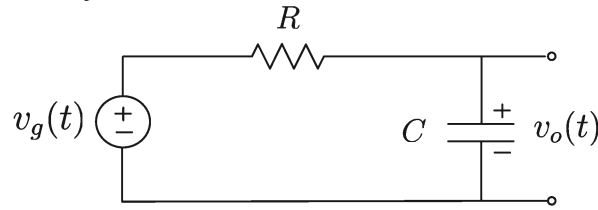


Figure 2. Low pass filter. Use $R = 1 \text{ k}\Omega$ and $C = 6.25 \text{ }\mu\text{F}$.

- Substitute the state space parameters into the $H_s(s)$ obtained using the unit step input, $1(kT)$. Obtain a simplified expression.
- Substitute the state space parameters into the $H_s(s)$ obtained using the sine wave input, $\cos(\omega_o kT)$. Obtain a simplified expression.
- Substitute the state space parameters into the $H_s(s)$ obtained using the impulse input, $\delta_0(kT)$. Obtain a simplified expression.

We next compare the magnitude frequency responses of each of the $H_s(s)$ transfer functions. We will use a sampling period of $T = \frac{2\pi}{5000}$ and $z = e^{j\omega T}$ with ω over the frequency range $10 \leq \omega \leq 10,000$. Use the Matlab command *semilogx* to plot the magnitude response of the following pairs of responses. The vertical axis should be in dB. The following Matlab commands should produce the desired results:

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w = logspace(1, 4, 200);
semilogx(w, 20*log10(abs(H)), w, 20*log10(abs(Hs)))
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- Plot the magnitude of $H(j\omega)$ (obtained using (4) with $s = j\omega$) and the magnitude of $T \times H_s(s)|_{z=e^{j\omega T}}$. Where H_s is that obtained using the unit step input, $1(kT)$, as explained in steps (4) and (8).
- Plot the magnitude of $H(j\omega)$ (obtained using (4) with $s = j\omega$) and the magnitude of $T \times H_s(s)|_{z=e^{j\omega T}}$. Where H_s is that obtained using the sine wave input, $\cos(\omega_o kT)$, as explained in steps (5) and (9).
- Plot the magnitude of $H(j\omega)$ (obtained using (4) with $s = j\omega$) and the magnitude of $T \times H_s(s)|_{z=e^{j\omega T}}$. Where H_s is that obtained using the impulse input, $\delta_0(kT)$ (Kronecker delta), as explained in steps (6) and (10).

Effect of aliasing:

- Determine or look up the expression for the spectrum of the three types of inputs: 1) unit step, 2) sine wave, and 3) Dirac delta.
- Comment on your results of Tasks 11, 12 and 13. Try and relate this to the spectra of the inputs. What has been the effect of sampling? Over what frequency range are the effects of sampling evident?

Report:

Write a report showing the derivation of all the results above. Email it to tymerski@ee.pdx.edu.