SOLUTION

Midterm #2

ECE 452/552: Control Systems Design II

- Closed book and closed notes, except as described below.
- One only ($8\frac{1}{2}'' \times 11''$) page of handwritten notes is permitted. (Written on both sides is OK).
- Calculators can be used.
- Scrap paper is not to be used. Show all work on the exam paper.

Student name: ___________________________________

Problem 1.
Use the Jury test to determine the stability of the following transfer function:

$$G(z) = \frac{1}{z^3 + 0.5z^2 - 1.34z + 0.24}$$

SOLUTION:

Define

$$P(z) = z^3 + 0.5z^2 - 1.34z + 0.24$$
$$= a_0 z^3 + a_1 z^2 + a_2 z + a_3 \qquad (a_0 > 0)$$

Then

$$a_0 = 1$$
$$a_1 = 0.5$$
$$a_2 = -1.34$$
$$a_3 = 0.24$$

The Jury stability conditions are

1. $|a_3| < a_0$

   This condition is satisfied.

2. $P(1) > 0$

   Since

   $$P(1) = 1 + 0.5 - 1.34 + 0.24 = 0.4 > 0$$

   the condition is satisfied.

3. $P(-1) < 0$

   Since

   $$P(-1) = -1 + 0.5 + 1.34 + 0.24 = 1.08 > 0$$

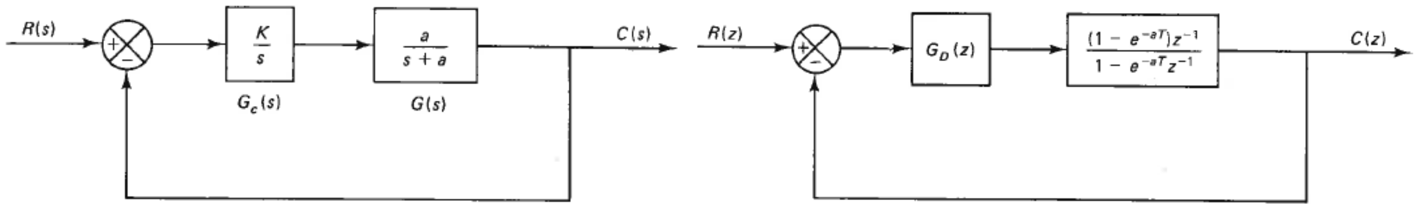
   the condition is not satisfied.

4. $|b_2| > |b_0|$

   Since condition (3) is not satisfied (the system is unstable), it is not necessary to test condition (4).

The conclusion is that the system is unstable.

Problem 2.
Consider the analog control system shown in Figure (a), where the first order plant $a/(s+a)$ is controlled by the analog integral controller K/s. The plant preceded by a zero order hold has a pulse transfer function as shown in Figure (b). Obtain the equivalent digital controller $G_D(z)$ based on the matched pole-zero mapping method. Hint: Be sure to match the static velocity error coefficients for the analog and digital systems.



(a) (b)

SOLUTION:

For the system of Figure (a), the static velocity error constant K_v is

$$K_v = \lim_{s\to 0}\left(s\frac{K}{s}\frac{a}{s+a}\right) = K$$

Next, we turn to the digital system of Figure (b).

The equivalent digital controller based on the matched pole-zero mapping method is

$$G_D(z) = \hat{K}\frac{z+1}{z-1} = \hat{K}\frac{1+z^{-1}}{1-z^{-1}}$$

where $\hat{K}$ is a constant. For a low-pass filter, such a constant is normally determined by the requirement that $G_D(1)$ and $G_c(0)$ be equal. In this particular case, both $G_D(1)$ and $G_c(0)$ become infinity and it is not possible to determine the unique value for $\hat{K}$. However, the constant $\hat{K}$ can be determined by the requirement that the analog control system and the equivalent digital control system have the same static velocity error constant. Thus, equating the static velocity error constant to K, we obtain

$$K_v = \lim_{z\to 1}\left[\frac{1-z^{-1}}{T}\hat{K}\frac{1+z^{-1}}{1-z^{-1}}\frac{(1-e^{-aT})z^{-1}}{1-e^{-aT}z^{-1}}\right] = K$$

or

$$\hat{K} = \frac{KT}{2}$$

Thus we have determined the constant $\hat{K}$. The equivalent digital controller based on the matched pole-zero mapping method is

$$G_D(z) = \frac{KT}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

Table of Laplace and Z-transforms

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	–	–	Kronecker delta $\delta_0(k)$ 1 $\quad k = 0$ 0 $\quad k \neq 0$	1
2.	–	–	$\delta_0(n-k)$ 1 $\quad n = k$ 0 $\quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$
18.	–	–	a^k	$\frac{1}{1-az^{-1}}$
19.	–	–	a^{k-1} $k = 1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.	–	–	ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.	–	–	k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.	–	–	k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.	–	–	k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.	–	–	$a^k\cos k\pi$	$\frac{1}{1+az^{-1}}$

$x(t) = 0$ for $t < 0$
$x(kT) = x(k) = 0$ for $k < 0$
Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$

TABLE 4–1 EQUIVALENT DISCRETE-TIME FILTERS FOR A CONTINUOUS-TIME FILTER $G(s) = a/(s + a)$

Mapping method	Mapping equation	Equivalent discrete-time filter for $G(s) = \frac{a}{s+a}$
Backward difference method	$s = \frac{1 - z^{-1}}{T}$	$G_D(z) = \frac{a}{\frac{1-z^{-1}}{T} + a}$
Forward difference method	$s = \frac{1 - z^{-1}}{Tz^{-1}}$	This method is not recommended, because the discrete-time equivalent may become unstable.
Bilinear transformation method	$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$	$G_D(z) = \frac{a}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + a}$
Bilinear transformation method with frequency prewarping	$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$ $\left(\omega_A = \frac{2}{T}\tan\frac{\omega_D T}{2}\right)$	$G_D(z) = \frac{\tan\frac{aT}{2}}{\frac{1-z^{-1}}{1+z^{-1}} + \tan\frac{aT}{2}}$
Impulse-invariance method	$G_D(z) = T\,\mathscr{Z}[G(s)]$	$G_D(z) = \frac{Ta}{1 - e^{-aT}z^{-1}}$
Step-invariance method	$G_D(z) = \mathscr{Z}\left[\frac{1 - e^{-Ts}}{s}G(s)\right]$	$G_D(z) = \frac{(1 - e^{-aT})z^{-1}}{1 - e^{-aT}z^{-1}}$
Matched pole-zero mapping method	A pole or zero at $s = -a$ is mapped to $z = e^{-aT}$. An infinite pole or zero is mapped to $z = -1$.	$G_D(z) = \frac{1 - e^{-aT}}{2}\frac{1 + z^{-1}}{1 - e^{-aT}z^{-1}}$