ECE452/552

HW #5

SOLUTION

Question (4) is from Ogata, "Discrete-Time Control Systems" 2nd ed., page 292.

1) Obtain the discrete-time equivalent of the following continuous-time filter by the use of

   (1) the backward difference method, and,

   (2) the impulse-invariance method:

$$G(s) = \frac{2}{(s+1)(s+2)}$$

   Assume that the sampling period is 0.1 sec, or $T = 0.1$.

2) Question 2 from the Midterm 2 practice exam:

   Use the matched pole-zero mapping method to determine the equivalent discrete-time filter for the following:

$$G(s) = \frac{s+a}{s(s+b)}$$

3) For the above questions (1) and (2) use Matlab (using the '*bode*' command) to compare the Bode plots of the continuous-time and discrete-time systems. For Question 2 use $a = 1, \ b = 10$ and sampling period, $T = 0.2$. So, for each of the three continuous-time and discrete-time frequency response pairs be sure to plot each pair on the same graph.

4) B-4-16 (from Ogata page 292)

SOLUTIONS:

Question 1:

$$G(s) = \frac{2}{(s+1)(s+2)}$$

1. Backward difference method: To obtain $G_D(z)$ we substitute

$$s = \frac{1 - z^{-1}}{T} = 10(1 - z^{-1})$$

into $G(s)$.

$$G_D(z) = \frac{2}{(10 - 10z^{-1} + 1)(10 - 10z^{-1} + 2)}$$

$$= \frac{0.01515}{1 - 1.7424z^{-1} + 0.7576z^{-2}}$$

2. Impulse invariance method:

$$G_D(z) = TG(z) = (0.1)\, \mathcal{Z}\left[\frac{2}{(s+1)(s+2)}\right]$$

$$= \frac{0.01722z^{-1}}{1 - 1.7235z^{-1} + 0.7408z^{-2}}$$

Question 2:

$$G(s) = \frac{s+a}{s(s+b)}$$

By use of the matched pole-zero mapping method, the equivalent discrete-time filter or digital filter can be obtained in the following form:

$$G_D(z) = K\frac{(z+1)(z - e^{-aT})}{(z-1)(z - e^{-bT})} = K\frac{(1 + z^{-1})(1 - e^{-aT}z^{-1})}{(1 - z^{-1})(1 - e^{-bT}z^{-1})}$$

Since both $G_D(1)$ and $G(0)$ approach infinity, constant K cannot be determined by equating $G_D(1)$ to $G(0)$. Therefore, we shall use the velocity error constant for determining constant K. For the analog filter

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{s+a}{s+b} = \frac{a}{b}$$

For the equivalent digital filter

$$K_v = \lim_{z \to 1} \frac{1 - z^{-1}}{T} G_D(z)$$

$$= \lim_{z \to 1} \frac{1 - z^{-1}}{T} K \frac{(1 + z^{-1})(1 - e^{-aT}z^{-1})}{(1 - z^{-1})(1 - e^{-bT}z^{-1})} = \frac{K}{T}\,\frac{2(1 - e^{-aT})}{1 - e^{-bT}}$$

Equating the K_v values obtained above, we get

$$K = \frac{aT}{2b} \frac{1 - e^{-bT}}{1 - e^{-aT}}$$

Hence, the equivalent digital filter $G_D(z)$ becomes as follows:

$$G_D(z) = \frac{aT}{2b} \frac{(1 - e^{-bT})}{(1 - e^{-aT})} \frac{(1 + z^{-1})(1 - e^{-aT}z^{-1})}{(1 - z^{-1})(1 - e^{-bT}z^{-1})}$$

Note that for sufficiently small values of T such that e^{-aT} and e^{-bT} can be approximated by

$$e^{-aT} \doteqdot 1 - aT \qquad \text{and} \qquad e^{-bT} \doteqdot 1 - bT$$

we have

$$K \doteqdot \frac{aT}{2b} \frac{1 - e^{-bT}}{1 - e^{-aT}} = \frac{T}{2}$$

Thus, the gain constant K becomes $\frac{1}{2}T$ for sufficiently small values of T.

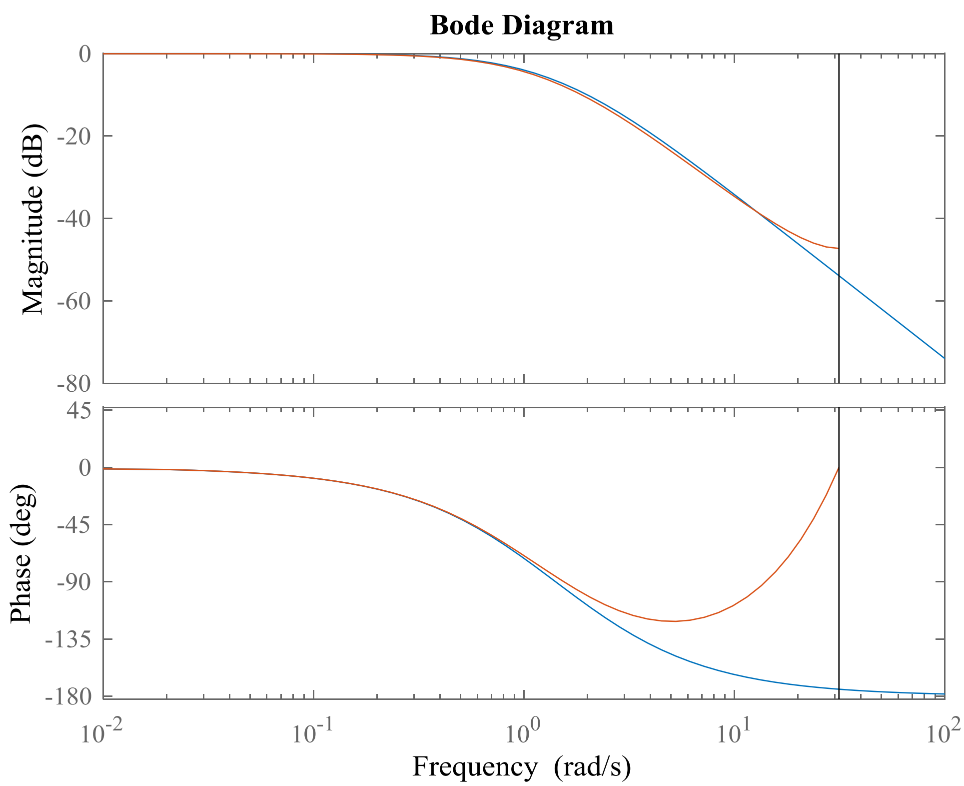
Question 3:

```
% Ogata 1st ed: Problem B-4-1
close all
clear

s = tf('s')

gs =2/((s+1)*(s+2))

z = tf('z', 0.1)

% Backward difference method
gz1 = 0.01515/(1-1.7424*z^(-1)+0.7576*z^(-2))
% figure
% bode(gs)
% figure
% bode(gz1)
figure
bode(gs,gz1)

% Impulse invariance method
gz2 = 0.01722*z^(-1)/(1-1.7235*z^(-1)+0.7408*z^(-2))
figure
bode(gs,gz2)
```

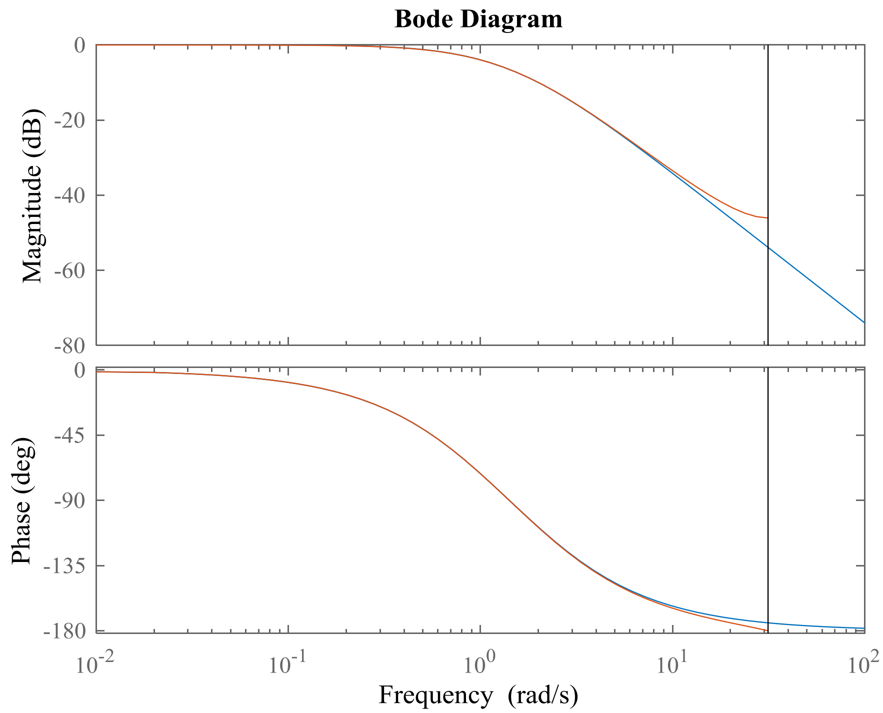
```
% Ogata 1st ed: Problem B-4-2

a = 1
b = 10

gs2 = (s+a)/(s*(s+b))

T = 0.2
z = tf('z', T)

% pole-zero mapping method
gz3 = a*T/(2*b) * (1-exp(-b*T))/(1-exp(-a*T)) * (1+z^(-1))/(1-z^(-1)) * ...
       (1-exp(-a*T)*z^(-1))/(1-exp(-b*T)*z^(-1))

figure
bode(gs2,gz3)
```

Plots from above Matlab code:

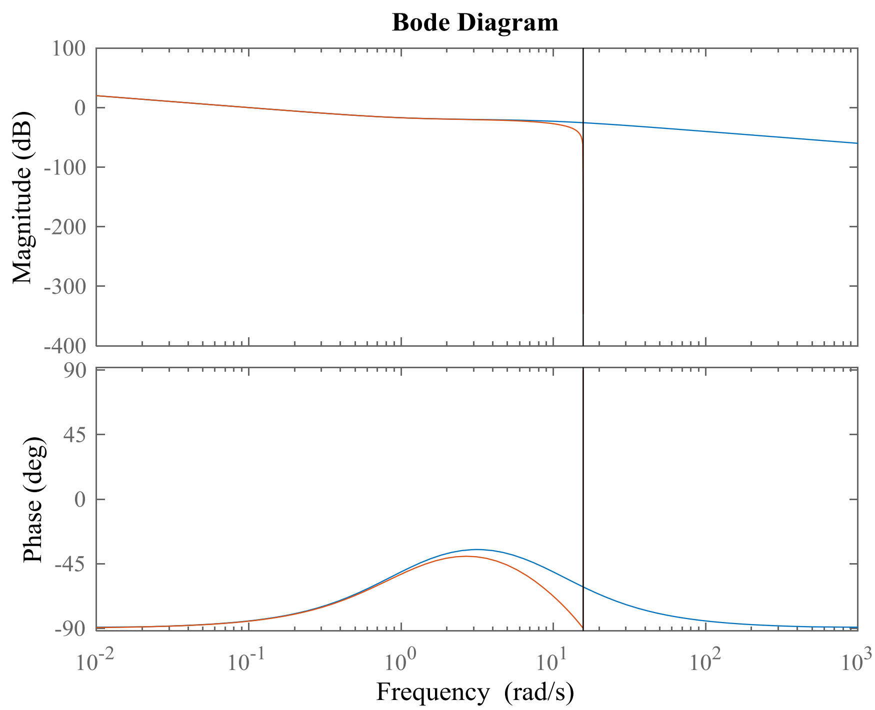
Question 1 using Backward difference method:



Question 1 using Impulse invariance method:



Question 2 using pole-zero mapping method:



Question 4:

B-4-16. For T = 0.1 sec, we have

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\,\frac{5}{(s + 1)(s + 2)}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{5}{s(s + 1)(s + 2)}\right]$$

$$= \frac{0.02263(z + 0.9061)}{(z - 0.9048)(z - 0.8187)}$$

Using the transformation

$$z = \frac{1 + \frac{1}{2}Tw}{1 - \frac{1}{2}Tw} = \frac{1 + 0.05w}{1 - 0.05w}$$

we have

$$G(w) = \frac{0.02263\left(\frac{1 + 0.05w}{1 - 0.05w} + 0.9061\right)}{\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.9048\right)\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.8187\right)}$$

$$= \frac{2.500\left(1 - \frac{1}{20}w\right)\left(1 + \frac{1}{406}w\right)}{(1 + w)\left(1 + \frac{1}{1.994}w\right)}$$

Notice that in order to have the static velocity error constant $K_v = 5\ \text{sec}^{-1}$, we need the controller $G_D(w)$ to include an integrator.

Using the conventional design approach, we find the following $G_D(w)$ will satisfy the requirements that the phase margin be 60°, the gain margin be not less than 12 db, and K_v be equal to 5 sec^{-1}.

$$G_D(w) = \frac{2}{w}\left(\frac{1 + \frac{1}{0.1}w}{1 + \frac{1}{0.01}w}\right)\left(\frac{1 + w}{1 + \frac{1}{10}w}\right)$$

Then the open-loop pulse transfer function becomes

$$G_D(w)G(w) = \frac{5}{w}\,\frac{\left(1 + \frac{1}{0.1}w\right)\left(1 - \frac{1}{20}w\right)\left(1 + \frac{1}{406}w\right)}{\left(1 + \frac{1}{0.01}w\right)\left(1 + \frac{1}{10}w\right)\left(1 + \frac{1}{1.994}w\right)}$$

From the Bode diagram of $G_D(w)G(w)$ (see next page), we find the phase margin to be approximately 60° and the gain margin to be approximately 22 dB. The gain crossover frequency is $\nu = 0.5$ rad/sec. The phase crossover frequency is $\nu = 3.5$ rad/sec.

Next, using the following transformation:

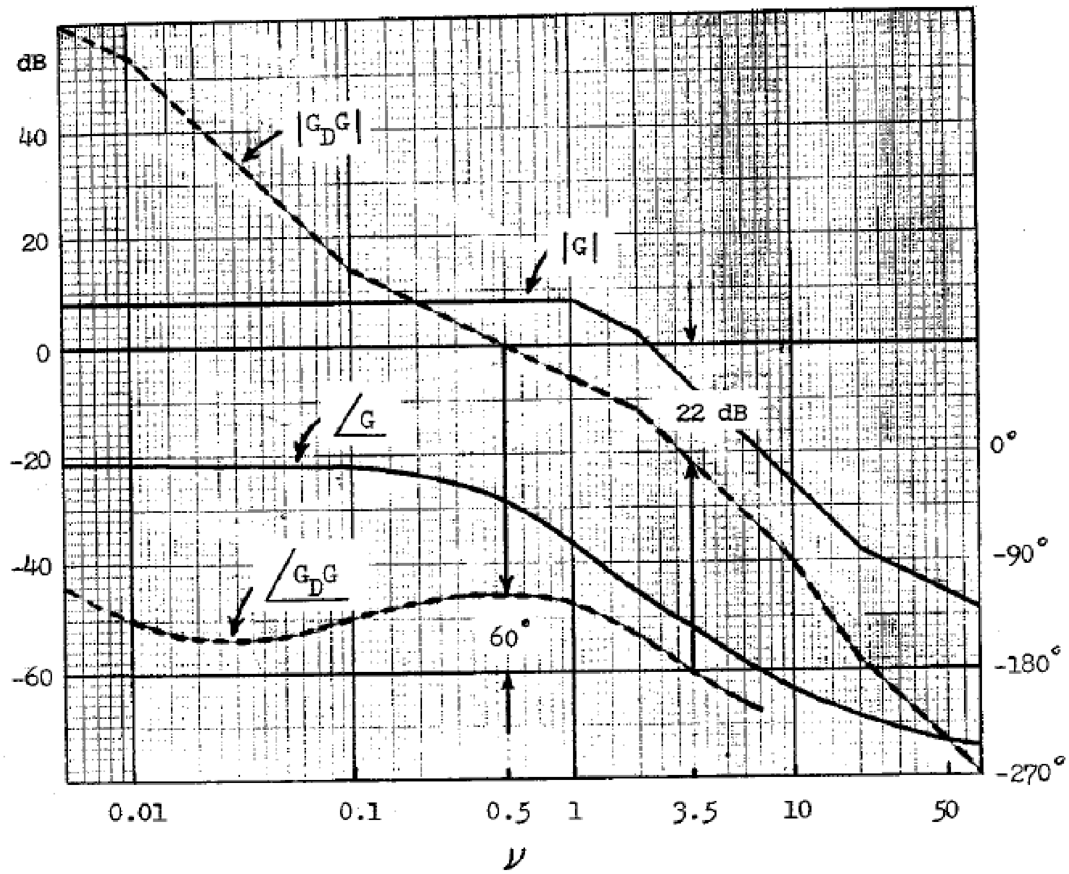
$$w = \frac{2}{0.1}\,\frac{z - 1}{z + 1} = 20\,\frac{z - 1}{z + 1}$$

we obtain $G_D(z)$ as follows:

$$G_D(z) = \frac{2\left(1 + \frac{1}{0.1}\,20\,\frac{z - 1}{z + 1}\right)\left(1 + 20\,\frac{z - 1}{z + 1}\right)}{20\left(\frac{z - 1}{z + 1}\right)\left(1 + \frac{1}{0.01}\,20\,\frac{z - 1}{z + 1}\right)\left(1 + \frac{1}{10}\,20\,\frac{z - 1}{z + 1}\right)}$$

$$= 0.07035\,\frac{(z + 1)(z - 0.9900)(z - 0.9048)}{(z - 1)(z - 0.9990)(z - 0.3333)}$$

The digital controller $G_D(z)$ defined by this last equation satisfies all the requirements of the problem and is, therefore, satisfactory.

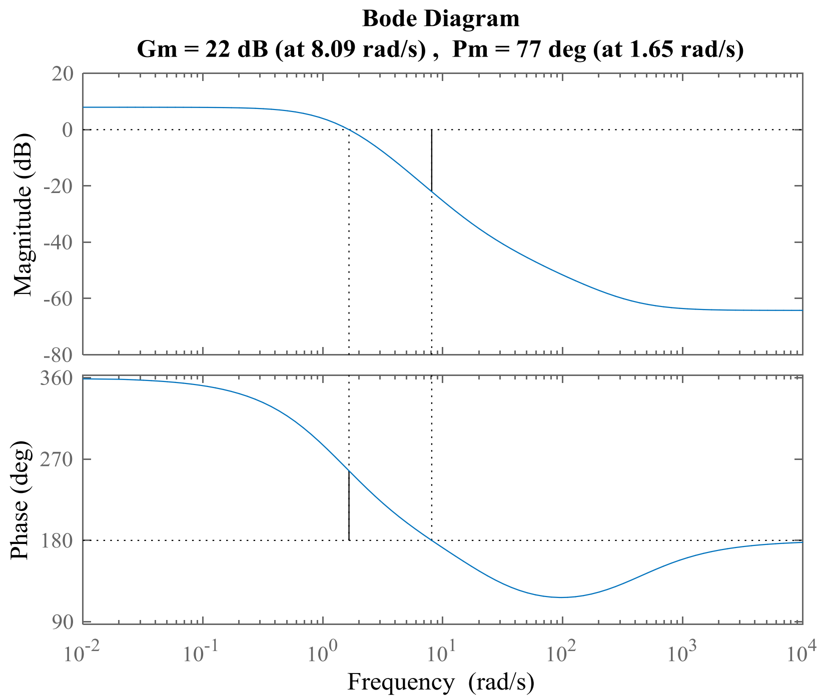


Matlab code for Question 4:

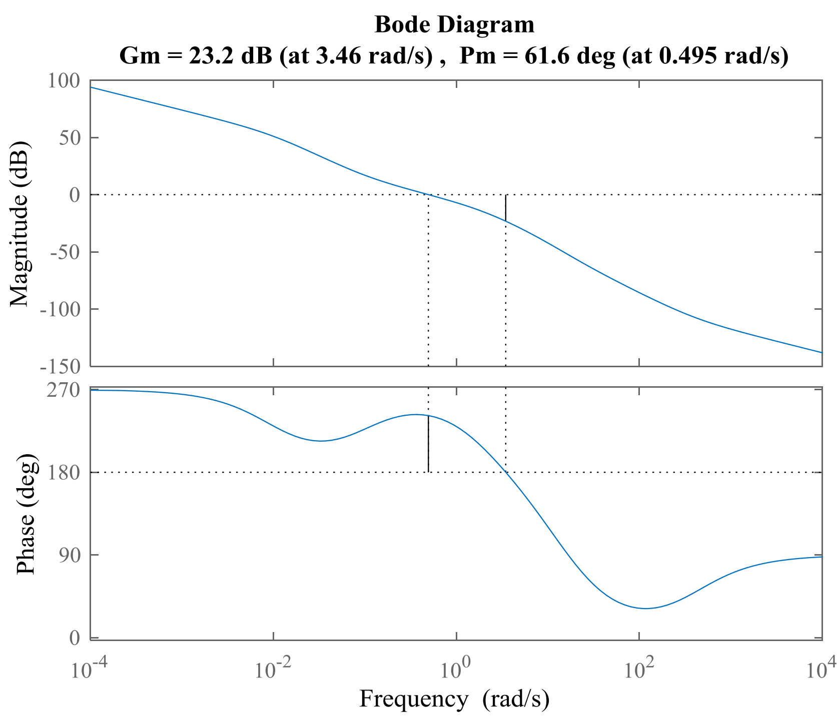
```
% Ogata Problem B-4-16
close all
clear

s = tf('s')

gw = 2.5*(1-s/20)*(1+s/406)/((1+s)*(1+s/1.994))
figure
margin(gw)

gd = 2/s *  (1+s/0.1)*(1+s)/((1+s/0.01)*(1+s/10))
figure
margin(gd*gw)

% another solution that has twice the bandwidth
gd = 0.4/s * (1+s)/((1+s/100)) * 5*(1+s/0.05)/(1+s/0.01)
figure
margin(gd*gw)
```

Results from the above Matlab code:

G(w) uncompensated:



Compensated loop gain, using the compensator derived above:



Compensated loop gain, using an alternative compensator: note the increased bandwidth