ECE452/552

HW #4

SOLUTION

Problems from Ogata, "Discrete-Time Control Systems" 2nd ed.

From pages 288-291

1) B-4-3
2) B-4-4
3) B-4-14

SOLUTIONS:

B-4-3.

$$\frac{Y(z)}{X(z)} = \frac{1}{z^3 + 0.5z^2 - 1.34z + 0.24}$$

Define

$$P(z) = z^3 + 0.5z^2 - 1.34z + 0.24$$
$$= a_0z^3 + a_1z^2 + a_2z + a_3$$

Then

$$a_0 = 1$$
$$a_1 = 0.5$$
$$a_2 = -1.34$$
$$a_3 = 0.24$$

The Jury stability conditions are

1. $|a_3| < a_0$

This condition is satisfied.

2. $P(1) > 0$

Since

$$P(1) = 1 + 0.5 - 1.34 + 0.24 = 0.4 > 0$$

the condition is satisfied.

3. $P(-1) < 0$

Since

$$P(-1) = -1 + 0.5 + 1.34 + 0.24 = 1.08 > 0$$

the condition is not satisfied.

4. $|b_2| > |b_0|$

Since condition (3) is not satisfied (the system is unstable), it is not necessary to test condition (4).

The conclusion is that the system is unstable.

B-4-4.

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-s}}{s} \frac{K}{s(s + 1)}\right] = (1 - z^{-1}) \mathcal{Z}\left[\frac{K}{s^2(s + 1)}\right]$$

$$= \frac{K\left[e^{-1}z^{-1} + (1 - 2e^{-1})z^{-2}\right]}{(1 - z^{-1})(1 - e^{-1}z^{-1})}$$

Hence

$$\frac{C(z)}{R(z)} = \frac{K\left[e^{-1}z^{-1} + (1 - 2e^{-1})z^{-2}\right]}{(1 - z^{-1})(1 - e^{-1}z^{-1}) + K\left[e^{-1}z^{-1} + (1 - 2e^{-1})z^{-2}\right]}$$

Noting that $e^{-1} = 0.3679$, the characteristic equation of the system becomes

$$z^2 - (1.3679 - 0.3679K)z + 0.3679 + 0.2642K = 0$$

Define

$$P(z) = z^2 - (1.3679 - 0.3679K)z + 0.3679 + 0.2642K$$

$$= a_0z^2 + a_1z + a_2$$

Then

$$a_0 = 1$$

$$a_1 = -1.3679 + 0.3679K$$

$$a_2 = 0.3679 + 0.2642K$$

For stability, we must have

$$|a_2| < a_0$$

$$P(1) > 0$$

$$P(-1) > 0$$

Therefore, we require

$$|0.3679 + 0.2642K| < 1$$

which yields

$$-5.1775 < K < 2.3925 \tag{1}$$

Also, from

$$P(1) = 1 - (1.3679 - 0.3679K) + 0.3679 + 0.2642K$$

$$= 0.6321K > 0$$

we obtain

$$K > 0 \tag{2}$$

and from

$$P(-1) = 1 + 1.3679 - 0.3679K + 0.3679 + 0.2642K$$

$$= 2.7358 - 0.1037K > 0$$

we have

$$26.38 > K \tag{3}$$

From Inequalities (1), (2), and (3), we obtain the range of gain K for stability to be

$$0 < K < 2.3925$$

B-4-14.

$$\hat{G}(z) = \mathcal{Z}\left[KG(s)\right] = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{K}{s(s+10)}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{K}{s^2(s+10)}\right]$$

$$= 0.01\,K\,\frac{0.2642z^{-2} + 0.3679z^{-1}}{(1 - z^{-1})(1 - 0.3679z^{-1})} = \frac{0.3679\,K(z + 0.7181)}{100(z - 1)(z - 0.3679)}$$

Since T = 0.1, we have

$$z = \frac{1 + \frac{1}{2}Tw}{1 - \frac{1}{2}Tw} = \frac{1 + 0.05w}{1 - 0.05w}$$

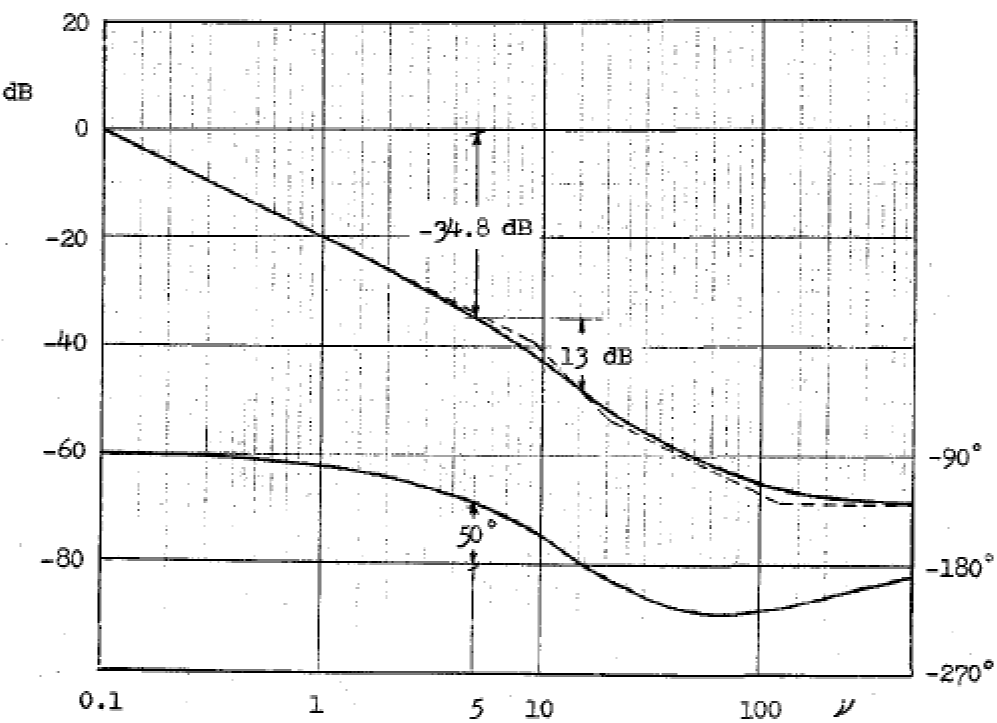
Then, $\hat{G}(w)$ becomes as follows:

$$\hat{G}(w) = \frac{0.3679\,K\left(\frac{1 + 0.05w}{1 - 0.05w} + 0.7181\right)}{100\left(\frac{1 + 0.05w}{1 - 0.05w} - 1\right)\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.3679\right)}$$

$$= \frac{0.1\,K(1 - 0.05w)(0.0082w + 1)}{w(0.1082w + 1)}$$

$$= \frac{0.1\,K\left(1 - \frac{1}{20}w\right)\left(\frac{1}{121.94}w + 1\right)}{w\left(\frac{1}{9.2421}w + 1\right)}$$

The Bode diagram of $\hat{G}(j\nu)$ with K = 1 is shown below. At ν = 5 rad/sec the phase angle is -130° and the magnitude $|\hat{G}(j5)|$ is -34.8 dB. Hence, to obtain the phase margin of 50°, we need to increase the magnitude of $\hat{G}(j5)$ by 34.8 dB. (That is, the entire magnitude curve must be raised by 34.8 dB.)



Thus, we require that the gain K be set such that

$$20 \log K \text{ dB} = 34.8 \text{ dB}$$

or

$$K = 55.0$$

With this gain value, the gain margin is 13 dB. The static velocity error constant K_v is obtained as

$$K_v = \lim_{w \to 0} w\hat{G}(w) = \lim_{w \to 0} w \frac{5.50(1 - 0.05w)(0.0082w + 1)}{w(0.1082w + 1)} = 5.50$$