

## ECE452/552

### HW #2

### SOLUTION

Problems from Ogata, "Discrete-Time Control Systems" 2<sup>nd</sup> ed.

From pages 166-167

- 1) B-3-2
  - 2) B-3-4 (use 1) residue method, and 2) partial fraction and table look up)
  - 3) B-3-6
  - 4) B-3-7 (find the analytical solution only, don't do the Matlab solution)
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### **SOLUTIONS:**

B-3-2. The differential equation for the circuit is

$$RC\dot{x} + x = e$$

For  $kT \leq t < (k+1)T$ ,  $e(t) = e(kT) = \text{constant}$ . Thus, we have

$$RC\dot{x} + x = e(kT) \quad kT \leq t < (k+1)T$$

Taking the Laplace transform of this last equation and considering  $t = kT$  to be the initial time, we obtain

$$RC \left[ sX(s) - x(kT) \right] + X(s) = \frac{e(kT)}{s}$$

or

$$X(s) = \frac{1}{RCs + 1} \left[ \frac{e(kT)}{s} + RCx(kT) \right] = \frac{e(kT)}{s} + \frac{RC [x(kT) - e(kT)]}{RCs + 1}$$

The inverse Laplace transform of this last equation is

$$x(t) = e(kT) + [x(kT) - e(kT)] e^{-\frac{1}{RC}(t - kT)} \quad kT \leq t < (k+1)T$$

Substituting  $t = (k+1)T - 0$  into this last equation, we obtain the desired difference equation as follows:

$$x((k+1)T-) = x(kT)e^{-\frac{T}{RC}} + (1 - e^{-\frac{T}{RC}})e(kT)$$

B-3-4.

1. Residue method:

$$\begin{aligned}
 X(z) &= \left[ \text{residue of } \frac{X(s)z}{z - e^{Ts}} \text{ at pole } s = -1 \right] \\
 &\quad + \left[ \text{residue of } \frac{X(s)z}{z - e^{Ts}} \text{ at pole } s = -2 \right] \\
 &= \lim_{s \rightarrow -1} \left[ (s+1) \frac{s+3}{(s+1)(s+2)} \frac{z}{z - e^{Ts}} \right] \\
 &\quad + \lim_{s \rightarrow -2} \left[ (s+2) \frac{s+3}{(s+1)(s+2)} \frac{z}{z - e^{Ts}} \right] \\
 &= \frac{2z}{z - e^{-T}} - \frac{z}{z - e^{-2T}} = \frac{2}{1 - e^{-T} z^{-1}} - \frac{1}{1 - e^{-2T} z^{-1}} \\
 &= \frac{1 + e^{-T}(1 - 2e^{-T})z^{-1}}{(1 - e^{-T}z^{-1})(1 - e^{-2T}z^{-1})}
 \end{aligned}$$

2. Method based on impulse response function:

$$X(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

The inverse Laplace transform of this equation gives

$$x(t) = 2e^{-t} - e^{-2t}$$

Hence

$$\begin{aligned}
 X(z) &= \frac{2}{1 - e^{-T} z^{-1}} - \frac{1}{1 - e^{-2T} z^{-1}} \\
 &= \frac{1 + e^{-T}(1 - 2e^{-T})z^{-1}}{(1 - e^{-T}z^{-1})(1 - e^{-2T}z^{-1})}
 \end{aligned}$$

### B-3-6

$$\begin{aligned}
 X(s) &= \frac{1 - e^{-Ts}}{s} \cdot \frac{1}{(s + a)^2} \\
 X(z) &= (1 - z^{-1}) \mathcal{Z} \left[ \frac{1}{s(s + a)^2} \right] \\
 &= (1 - z^{-1}) \mathcal{Z} \left[ \frac{\frac{1}{a^2} \frac{1}{s}}{a^2} - \frac{1}{a^2} \frac{1}{s + a} - \frac{1}{a} \frac{1}{(s + a)^2} \right] \\
 &= (1 - z^{-1}) \left[ \frac{\frac{1}{a^2} \frac{1}{1 - z^{-1}}}{1 - e^{-aT} z^{-1}} - \frac{1}{a^2} \frac{1}{1 - e^{-aT} z^{-1}} \right. \\
 &\quad \left. - \frac{1}{a} \frac{T a^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^2} \right] \\
 &= \frac{1}{a^2} \frac{(1 - e^{-aT}) z^{-1}}{1 - e^{-aT} z^{-1}} - \frac{1}{a} \frac{(1 - z^{-1}) T e^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^2}
 \end{aligned}$$

### B-3-7.

$$y(k+1) + 0.5y(k) = x(k), \quad y(0) = 0$$

The z transform of this equation is

$$zY(z) - zy(0) + 0.5Y(z) = X(z) = \frac{z}{z - 1}$$

Solving this equation for Y(z), noting that y(0) = 0, we obtain

$$Y(z) = \frac{z}{(z + 0.5)(z - 1)} = -\frac{1}{1.5} \frac{z}{z + 0.5} + \frac{1}{1.5} \frac{z}{z - 1}$$

Hence

$$y(k) = -\frac{1}{1.5} (-0.5)^k + \frac{1}{1.5} = \frac{2}{3} \left[ 1 - (-0.5)^k \right]$$