ECE452/552

HW #1

SOLUTION

Problems from Ogata, "Discrete-Time Control Systems" 2nd ed.

From pages 70 -73

1) B-2-1
2) B-2-7
3) B-2-9 (use the partial fractions method only)
4) B-2-11 (use the inversion integral method only)
5) B-2-17

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1) B-2-1

$$X(z) = \mathcal{Z}\left[\frac{1}{a}\left(1 - e^{-at}\right)\right] = \frac{1}{a}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT}z^{-1}}\right)$$

$$= \frac{1}{a}\,\frac{z^{-1}(1 - e^{-aT})}{(1 - z^{-1})(1 - e^{-aT}z^{-1})}$$

2) B-2-7

From Figure 2-8 we have

$x(0) = 0, \quad x(1) = 0, \quad x(2) = 0, \quad x(3) = \frac{1}{3}$

$x(4) = \frac{2}{3}, \quad x(k) = 1 \quad$ for $k = 5, 6, 7, \ldots$

Then

$$X(z) = \sum_{k=0}^{\infty} x(k)\, z^{-k}$$

$$= \frac{1}{3} z^{-3} + \frac{2}{3} z^{-4} + z^{-5} + z^{-6} + z^{-7} + \cdots$$

$$= \frac{1}{3} (z^{-3} + 2z^{-4}) + \frac{z^{-5}}{1 - z^{-1}}$$

$$= \frac{1}{3} \frac{z^{-3} + z^{-4} + z^{-5}}{1 - z^{-1}}$$

3) B-2-9

<u>Partial-fraction-expansion method:</u>

$$X(z) = \frac{z^{-1}(0.5 - z^{-1})}{(1 - 0.5z^{-1})(1 - 0.8z^{-1})^2} = \frac{z(0.5z - 1)}{(z - 0.5)(z - 0.8)^2}$$

Hence,

$$\frac{X(z)}{z} = -\frac{8.3333}{z - 0.5} + \frac{8.3333}{z - 0.8} - \frac{2}{(z - 0.8)^2}$$

or

$$X(z) = -\frac{8.3333}{1 - 0.5z^{-1}} + \frac{8.3333}{1 - 0.8z^{-1}} - \frac{2z^{-1}}{(1 - 0.8z^{-1})^2}$$

Thus,

$$x(k) = -8.3333(0.5)^k + 8.3333(0.8)^k - 2k(0.8)^{k-1}, \quad k = 0, 1, 2, \ldots$$

4) B-2-11

<u>Inversion integral method:</u>

$$X(z) = \frac{1 + z^{-1} - z^{-2}}{1 - z^{-1}} = \frac{z^2 + z - 1}{(z - 1)z}$$

Hence

$$X(z)z^{k-1} = \frac{(z^2 + z - 1)z^{k-1}}{(z - 1)z}$$

<u>For k = 0:</u>

$$X(z)z^{k-1} = \frac{z^2 + z - 1}{(z - 1)z^2}$$

Thus,

$$x(0) = \left[\text{residue of } \frac{z^2 + z - 1}{(z - 1)z^2} \text{ at pole } z = 1\right]$$

$$+ \left[\text{residue of } \frac{z^2 + z - 1}{(z - 1)z^2} \text{ at double pole } z = 0\right]$$

$$= \lim_{z \to 1} \left[(z - 1)\frac{z^2 + z - 1}{(z - 1)z^2}\right]$$

$$+ \frac{1}{(2 - 1)!} \lim_{z \to 0} \frac{d}{dz}\left[z^2 \frac{z^2 + z - 1}{(z - 1)z^2}\right] = 1 + 0 = 1$$

<u>For k = 1:</u>

$$X(z)z^{k-1} = \frac{z^2 + z - 1}{(z - 1)z}$$

$$x(1) = \left[\text{residue of } \frac{z^2 + z - 1}{(z - 1)z} \text{ at pole } z = 1\right]$$

$$+ \left[\text{residue of } \frac{z^2 + z - 1}{(z - 1)z} \text{ at pole } z = 0\right]$$

$$= \lim_{z \to 1} \left[(z - 1)\frac{z^2 + z - 1}{(z - 1)z}\right] + \lim_{z \to 0} \left[z \frac{z^2 + z - 1}{(z - 1)z}\right]$$

$$= 1 + 1 = 2$$

For $k = 2, 3, 4, \ldots$:

$$X(z)z^{k-1} = \frac{(z^2 + z - 1)z^{k-2}}{z - 1}$$

Hence

$$x(k) = \left[\text{residue of } \frac{(z^2 + z - 1)z^{k-2}}{z - 1} \text{ at pole } z = 1\right]$$

$$= \lim_{z \to 1} \left[(z - 1)\frac{(z^2 + z - 1)z^{k-2}}{z - 1}\right] = 1$$

Therefore,

$$x(0) = 1$$

$$x(1) = 2$$

$$x(k) = 1 \qquad \text{for } k = 2, 3, 4, \ldots$$

5) B-2-17

$$x(k + 2) - x(k + 1) + 0.25x(k) = u(k + 2)$$

The z transform of this difference equation is

$$\left[z^2X(z) - z^2x(0) - zx(1)\right] - \left[zX(z) - zx(0)\right] + 0.25X(z)$$
$$= z^2U(z) - z^2u(0) - zu(1)$$

Substituting the initial data into this last equation, we get

$$(z^2 - z + 0.25)X(z) = \frac{z^3}{z - 1}$$

or

$$X(z) = \frac{z^3}{(z - 1)(z^2 - z + 0.25)}$$

$$= \frac{4z}{z - 1} - \frac{3z}{z - 0.5} - \frac{0.5z}{(z - 0.5)^2}$$

Hence

$$x(k) = 4 - (3 + k)(0.5)^k \qquad \text{for } k = 0, 1, 2, \ldots$$

The following shows the Matlab code used to show output response using 3 different methods:

1) `Computational solution with MATLAB using difference equation`
2) `Computational solution with MATLAB using the 'filter' command`
3) `analytical solution derived above`

```
% Ogata B-2-17
format compact
close all
clear

% Computational solution with MATLAB using difference equation
u = ones(1,41);
x(1) = 1;
x(2) = 2;
for n = 1:39
    x(n+2) = x(n+1)-0.25*x(n)+u(n+2);
end

k=0:40;
plot(k,x,'o')
grid
title('Unit Step Response')
xlabel('k')
ylabel('y(k)')
hold on

% Computational solution with MATLAB
num = [1 0 0];
den = [1 -1 0.25];

y = filter(num, den, u);
plot(k,y,'bx')

% analytical solution
num = [1 0 0];
den = conv([1, -1], [1, -1, 0.25]);
[r, p, kr] = residue(num,den)

% r =
%     4.0000
%    -3.0000
%    -0.5000
% p =
%     1.0000
%     0.5000
%     0.5000
% kr =
%      []

z = r(1)*(p(1).^k) + r(2)*(p(2).^k) + r(3)*k .*(p(3).^(k-1));
%z = r(1)*(p(1).^k) + r(2)*(p(2).^k) - k .*(p(3).^k);
plot(k,z,'r+')
hold off

% compare the results of the 3 approaches
yy = [x;y;z]
```

The 3 sets of results are the same:

```
yy =

  Columns 1 through 14

    1.0000    2.0000    2.7500    3.2500    3.5625    3.7500    3.8594    3.9219    3.9570    3.9766    3.9873    3.9932    3.9963    3.9980

    1.0000    2.0000    2.7500    3.2500    3.5625    3.7500    3.8594    3.9219    3.9570    3.9766    3.9873    3.9932    3.9963    3.9980

    1.0000    2.0000    2.7500    3.2500    3.5625    3.7500    3.8594    3.9219    3.9570    3.9766    3.9873    3.9932    3.9963    3.9980

  Columns 15 through 28

    3.9990    3.9995    3.9997    3.9998    3.9999    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000

    3.9990    3.9995    3.9997    3.9998    3.9999    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000

    3.9990    3.9995    3.9997    3.9998    3.9999    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000

  Columns 29 through 41

    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000

    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000

    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000    4.0000
```