ECE 451: Control Systems Design I – Notes

TF:

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$$

TF →SS: for proper SISO *T(s):*

Phase-Variable Form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & \vdots \\ \vdots & \ldots & 0 & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & \ldots & \ldots & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [b_n \quad b_{n-1} \quad \ldots \quad b_1] \qquad D = [0]$$

Dual Phase-Variable Form:

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & \ldots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 \\ -a_n & 0 & \ldots & \ldots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$C = [1 \quad 0 \quad \ldots \quad 0] \qquad D = [0]$$

SS →TF: $T(s) = C(sI-A)^{-1}B + D$

Diagonalization of SS: using only a state transformation:

$$\bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \qquad \bar{B} = P^{-1}B \qquad \bar{C} = CP \qquad \bar{D} = D$$

where: $P = [eigvector(\lambda_1) \quad \cdots \quad eigvector(\lambda_n)]$ and: $|s\lambda - A| = 0$

Solution of the state equation – time response of systems:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau = e^{A(t-t_0)}x(t_0) + A^{-1}\left[e^{A(t-t_0)} - I\right]BU$$

where: $u(t) = U$ is a constant; $e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$

M_C and M_O Matrices:

$$M_C = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B] \qquad M_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Tracking problems: $e_{ss} = r - y_{ss}$ $\quad \bar{N} = \frac{-1}{C(A-BK)^{-1}B}$ $\quad \begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}u + \begin{bmatrix} 0 \\ 1 \end{bmatrix}r$

Controller design, using Ackerman's formula:

$$k = [0 \quad \ldots \quad 0 \quad 1]M_C^{-1}\Delta_d(A)$$

where: $\Delta_d(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_n I$

Full Order Observer (FOE): $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

Error: $\tilde{x} = x - \hat{x}$ $\qquad \dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = (A - LC)\tilde{x}$

Observer design, using Ackerman's formula:

$$L = \Delta_e(A)M_O^{-1}\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

Control using Observers:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Seperation Principle: $\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$

FOE Compensator Transfer Function: $\frac{U(s)}{Y(s)} = -K(sI - A + BK + LC)^{-1}L$

Reduced Order Observer (ROE): $\begin{bmatrix} \dot{x_m} \\ \dot{x_u} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_m \\ x_u \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$

$\dot{\widetilde{x_u}} = (A_{22} - LA_{12})\widetilde{x_u}$

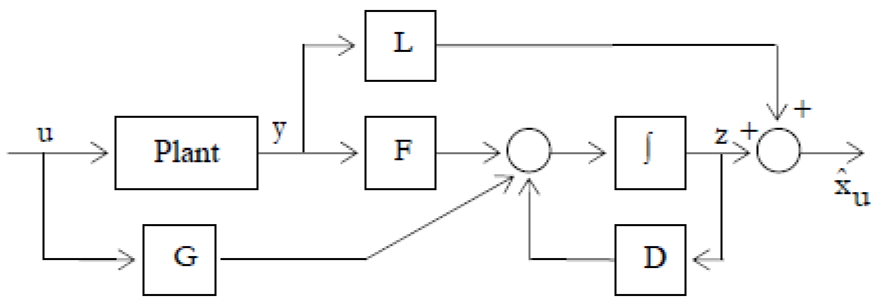
$\dot{z} = Dz + Fy + Gu$

$\widehat{x_u} = z + Ly$

$D = A_{22} - LA_{12}$

$F = DL + A_{21} - LA_{11}$

$G = B_2 - LB_1$



ROE Compensator Transfer Function: $\frac{U(s)}{Y(s)} = C'(sI - A')^{-1}B' + D'$

$A' = D - GK_2$ $\qquad$ $B' = F - GK_1 - GK_2L$

$C' = -K_2$ $\qquad$ $D' = -(K_1 + K_2L)$

LQR: Find a control function *u(t)* that minimizes the cost function *J*:

$J = \frac{1}{2}\int_0^\infty (x^T Qx + u^T Ru)\, dt$ subject to: $\dot{x} = Ax + Bu$

$R > 0$ is positive definite and $Q \geq 0$ is positive semi definite

The solution exists if: (1). (A,B) is controllable (2). $R > 0$ (3). $Q = C_q^T C_q$, where (C_q, A) is observable

Solution: **Algebraic Riccati Equation (ARE):**

$A^T P + PA + Q - PBR^{-1}B^T P = 0$ where P is a positive definite solution,

$u = -Kx$ where: $K = R^{-1}B^T P$

LQE: $\dot{x} = Ax + Bu + w$

$y = Cx + v$

Solution: $L = \sum C^T R_O^{-1}$ where $\sum$ is the positive definite solution of:

$A\sum + \sum A^T + Q_O - \sum C^T R_O^{-1} C \sum = 0$ with $Q_O \geq 0$ and $R_O > 0$

LQG: You can recover some properties of LQR if: (1) the plant is minimum phase

(2) $R_O = 1$ and $Q_O = q^2 BB^T$

LQG/LQR: Recover LQR properties by using Q_O and R_O as tuning parameters.