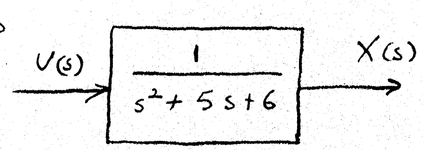
EE451 / EE551

HW SYSTEM RESPONSE

PROBLEM

Consider the system shown in block diagram form. →



a) Find the state-space representation of the system, using the following state variables.

$$x_1 = x$$

$$x_2 = \dot{x}$$

b) Find the state transition matrix $\phi(t)$ associated with the representation of part (a).

c) Verify that for the ϕ obtained in part (b), we have

$$\dot{\phi} = A\phi$$

d) Find the free (no input) response of the system for

$$x_1(0) = 0$$

$$x_2(0) = 1$$

e) Find the system's response for the same initial conditions as in part (d) with $u(t) = 1$.

f) Find a state representation of the system such that the matrix A is diagonal.

g) Verify the results of part (d) and (e) using the representation of part (f).

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SOLUTION

(a) The transfer function of the system is given by

$$\frac{x(s)}{u(s)} = \frac{1}{s^2 + 5s + 6}$$

Thus the differential equation form is given by

$$\ddot{x} + 5\dot{x} + 6x = u$$

Let

$$x_1 = x$$
$$x_2 = \dot{x}$$

Thus

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \ddot{x} = -6x_1 - 5x_2 + u$$

As a result,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Thus the matrix **A** is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

(b) We have

$$\Phi^{-1}(s) = [sI - \mathbf{A}] = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$

Thus

$$\Phi(s) = \frac{1}{\Delta} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

The determinant is given by

$$\Delta = s^2 + 5s + 6 = (s+3)(s+2)$$

We now have

$$\phi_{11}(s) = \frac{s+5}{\Delta} = \frac{3}{s+2} - \frac{2}{s+3}$$
$$\phi_{12}(s) = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$$
$$\phi_{21}(s) = \frac{-6}{(s+2)(s+3)} = \frac{6}{s+3} - \frac{6}{s+2}$$
$$\phi_{22}(s) = \frac{s}{(s+2)(s+3)} = \frac{3}{s+3} - \frac{2}{s+2}$$

The inverse Laplace transform gives us

THIS IS $\phi(t)$ IN CLASS NOTES.

$$\phi(t,0) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ 6e^{-3t} - 6e^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix}$$

(c) We have by differentiation

$$\dot{\phi}(t,0) = \begin{bmatrix} 6e^{-3t} - 6e^{-2t} & 3e^{-3t} - 2e^{-2t} \\ 12e^{-2t} - 18e^{-3t} & 4e^{-2t} - 9e^{-3t} \end{bmatrix}$$

We also compute

$$\mathbf{A}\phi = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ 6e^{-3t} - 6e^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix}$$
$$= \begin{bmatrix} 6e^{-3t} - 6e^{-2t} & 3e^{-3t} - 2e^{-2t} \\ 12e^{-2t} - 18e^{-3t} & 4e^{-2t} - 9e^{-3t} \end{bmatrix}$$

Thus

$$\mathbf{A}\phi = \dot{\phi}$$

(d) We have for the force-free response,

$$\mathbf{x}(t) = \phi(t,0)\mathbf{x}(0)$$
$$\mathbf{x}(t) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Thus with

$$x_1(0) = 0$$
$$x_2(0) = 1$$

we get

$$\mathbf{x}(t) = \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} e^{-2t} & -e^{-3t} \\ 3e^{-3t} & -2e^{-2t} \end{bmatrix}$$

(e) With a forcing function we have

$$\mathbf{x}(t) = \phi(t,0)\mathbf{x}(0) + \int_0^t \phi(t,\tau)\mathbf{B}(\tau)u(\tau)\,d\tau$$

Now the integrand is given by

$$\boldsymbol{\phi}(t,\tau)\mathbf{B}(\tau)u(\tau) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(\tau)$$

$$= \begin{bmatrix} \phi_{12}(t,\tau) \\ \phi_{22}(t,\tau) \end{bmatrix}u(\tau)$$

$$= \begin{bmatrix} e^{-2(t-\tau)} & -e^{-3(t-\tau)} \\ 3e^{-3(t-\tau)} & -2e^{2(t-\tau)} \end{bmatrix}$$

Thus

$$\int_0^t \boldsymbol{\phi}(t,\tau)\mathbf{B}(\tau)u(\tau)\,d\tau = \begin{bmatrix} e^{-2t}\left(\int_0^t e^{2\tau}\,d\tau\right) - e^{-3t}\left(\int_0^t e^{3\tau}\,d\tau\right) \\ 3e^{-3t}\int_0^t e^{3\tau}\,d\tau - 2e^{-2t}\int_0^t e^{2\tau}\,d\tau \end{bmatrix}$$

see later for alternative method to obtain this

$$= \begin{bmatrix} \frac{1}{6} - 0.5e^{-2t} + \frac{1}{3}e^{-3t} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

As a result,

$$x_1(t) = (e^{-2t} - e^{-3t}) + \tfrac{1}{6} - 0.5e^{-2t} + \tfrac{1}{3}e^{-3t}$$
$$= \tfrac{1}{6} + 0.5e^{-2t} - \tfrac{2}{3}e^{-3t}$$
$$x_2(t) = (3e^{-3t} - 2e^{-2t}) + e^{-2t} - e^{-3t}$$
$$= 2e^{-3t} - e^{-2t}$$

(f) Let

$$\hat{\mathbf{x}} = \mathbf{T}\mathbf{x}$$

Differentiating, we get

$$\dot{\hat{\mathbf{x}}} = \mathbf{T}\dot{\mathbf{x}} = \mathbf{T}\mathbf{A}\mathbf{x} + \mathbf{T}\mathbf{B}\mathbf{u}$$

or

$$\dot{\hat{\mathbf{x}}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\hat{\mathbf{x}} + \mathbf{T}\mathbf{B}\mathbf{u}$$

Thus

$$\dot{\hat{\mathbf{x}}} = \tilde{\mathbf{A}}\hat{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u}$$

where

$$\tilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$$

or

$$\tilde{\mathbf{A}}\mathbf{T} = \mathbf{T}\mathbf{A}$$

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

Expanding, we have

$$a_{11}t_{11} = -6t_{12} \quad (1)$$
$$a_{11}t_{12} = t_{11} - 5t_{12} \quad (2)$$
$$a_{22}t_{21} = -6t_{22} \quad (3)$$
$$a_{22}t_{22} = t_{21} - 5t_{22} \quad (4)$$

Substitution of Eq. (2) in (1) gives

$$a_{11}(a_{11}t_{12} + 5t_{12}) + 6t_{12} = 0$$

or

$$a_{11}^2 + 5a_{11} + 6 = 0$$
$$(a_{11} + 3)(a_{11} + 2) = 0$$

Thus we take $a_{11} = -2$. Substitution of Eq. (4) in (3) gives

$$a_{22}(a_{22}t_{22} + 5t_{22}) = -6t_{22}$$
$$(a_{22} + 3)(a_{22} + 2) = 0$$

Take

$$a_{22} = -3$$

Thus

$$-2t_{11} = -6t_{12} \qquad t_{11} = 3t_{12}$$
$$-3t_{21} = -6t_{22} \qquad t_{21} = 2t_{22}$$

Take $t_{12} = 1$ to obtain $t_{11} = 3$; similarly, $t_{22} = 2$ to obtain $t_{21} = 4$. Thus the transform matrix $\mathbf{T}$ is obtained as

$$\mathbf{T} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

The inverse of $\mathbf{T}$ is obtained as

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & \frac{-1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$$

Since

$$\mathbf{x} = \mathbf{T}^{-1}\hat{\mathbf{x}}$$

we conclude that

$$x_1 = \hat{x}_1 - 0.5\hat{x}_2$$
$$x_2 = -2\hat{x}_1 + 1.5\hat{x}_2$$

(f) We have

$$\tilde{\mathbf{A}} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

The associated state transition matrix is thus given by

$$\tilde{\boldsymbol{\phi}} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

(g) For part (d) we have

$$\hat{x}_1 = 3x_1 + x_2$$
$$\hat{x}_2 = 4x_1 + 2x_2$$

Thus the initial conditions are

$$\hat{x}_1(0) = 3(0) + 1 = 1$$
$$\hat{x}_2(0) = 4(0) + 2 = 2$$

Thus the free response is obtained as

$$\hat{\mathbf{x}}_f(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ 2e^{-3t} \end{bmatrix}$$

Expanding, we get

$$\hat{x}_{1_f}(t) = e^{-2t}$$
$$\hat{x}_{2_f}(t) = 2e^{-3t}$$

Thus for the original system form, we have

$$x_{1_f}(t) = e^{-2t} - e^{-3t}$$
$$x_{2_f}(t) = -2e^{-2t} + 3e^{-3t}$$

For part (e)

$$\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_f(t) + \int_0^t \tilde{\boldsymbol{\phi}}(t,\tau)\tilde{\mathbf{B}}(\tau)u(\tau)\,d\tau$$

$$\tilde{\boldsymbol{\phi}}(t,\tau)\tilde{\mathbf{B}}(\tau)u(\tau) = \begin{bmatrix} e^{-2(t-\tau)} & 0 \\ 0 & e^{-3(t-\tau)} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(\tau)$$
$$= \begin{bmatrix} e^{-2(t-\tau)} \\ 2e^{-3(t-\tau)} \end{bmatrix}u(\tau)$$

$$\int_0^t \tilde{\boldsymbol{\phi}}(t,\tau)\tilde{\mathbf{B}}(\tau)u(\tau)\,d\tau = \begin{bmatrix} e^{-2t}\left(\dfrac{e^{2t}}{2}\right)_0^t \\ 2e^{-3t}\left(\dfrac{e^{3\tau}}{3}\right)_0^t \end{bmatrix}$$
$$= \begin{bmatrix} 0.5e^{-2t}(e^{2t} - 1) \\ \dfrac{2e^{-3t}}{3}(e^{3t} - 1) \end{bmatrix}$$

Thus

$$\hat{x}_1(t) = e^{-2t} + 0.5(1 - e^{-2t})$$
$$= 0.5 + 0.5e^{-2t}$$
$$\hat{x}_2(t) = 2e^{-3t} + \tfrac{2}{3}(1 - e^{-3t})$$
$$= \tfrac{2}{3} + \tfrac{4}{3}e^{-3t}$$

Thus

$$x_1(t) = \hat{x}_1(t) - 0.5\hat{x}_2(t)$$
$$= 0.5 + 0.5e^{-2t} - \tfrac{1}{3} - \tfrac{2}{3}e^{-3t}$$
$$= \tfrac{1}{6} + \tfrac{1}{2}e^{-2t} - \tfrac{2}{3}e^{-3t}$$
$$x_2(t) = -2\hat{x}_1(t) + 1.5\hat{x}_2(t)$$
$$= -1 - e^{-2t} + 1 + 2e^{-3t}$$
$$= 2e^{-3t} - e^{-2t}$$

Clearly, we obtain the same results.

ALTERNATIVE SOLUTION TO (e)

The forced response can be found from

$$x_f(t) = A^{-1}\left[e^{A(t-t_0)} - I\right] B U$$

when u is constant

$$\Rightarrow \quad x_f(t) = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 3e^{-2t} - 2e^{-3t} - 1 & e^{-2t} - e^{-3t} \\ 6e^{-3t} - 6e^{-2t} & 3e^{-3t} - 2e^{-2t} - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1$$

$$= \frac{1}{6}\begin{bmatrix} -5 & -1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} e^{-2t} - e^{-3t} \\ 3e^{-3t} - 2e^{-2t} - 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6}\left(-5e^{-2t} + 5e^{-3t} - 3e^{-3t} + 2e^{-2t} + 1\right) \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

$$= \begin{bmatrix} -0.5e^{-2t} + \frac{1}{3}e^{-3t} + \frac{1}{6} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

which corresponds to that previously obtained using the other method