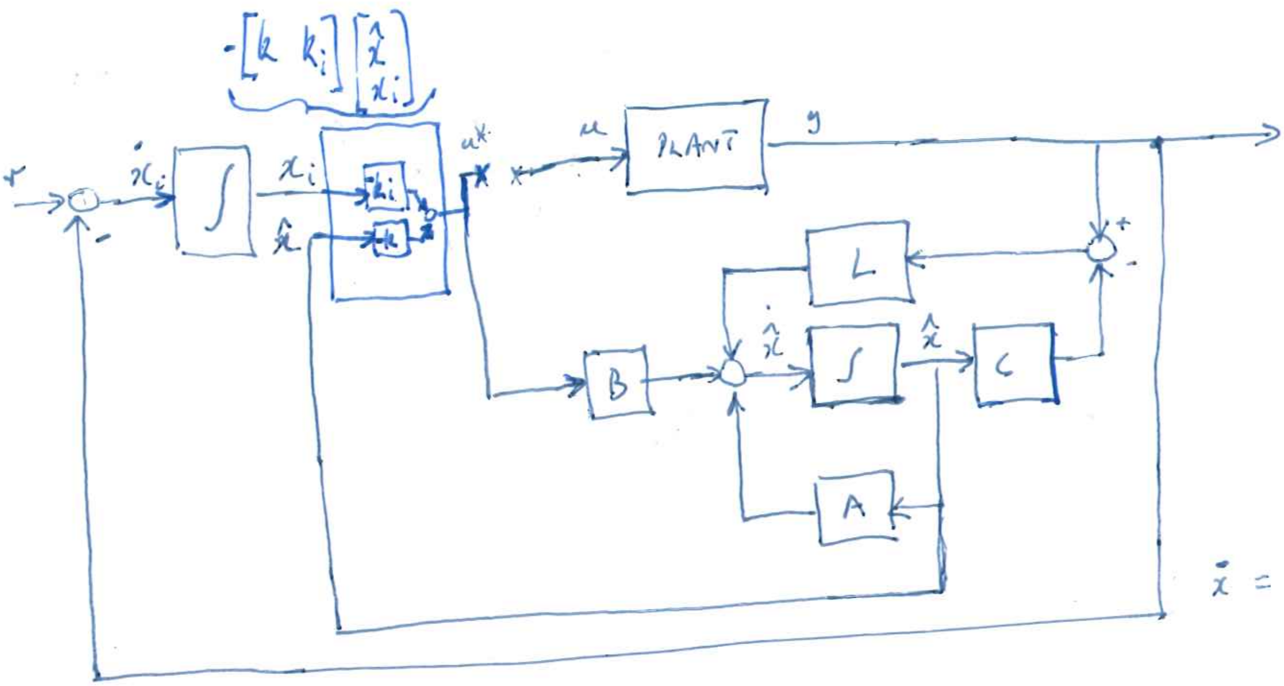
INTEGRAL WITH FOE



$\bar{x} = \begin{bmatrix}\hat{x}\\ x_i\end{bmatrix}$

PLANT :

$$\dot{x} = Ax + Bu$$
$$y = Cx.$$

FOE :

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$$

INTEGRAL CONTROL :

$$\dot{x}_i = -y + r$$

CONTROL LAW :

$$u = -\bar{k}\,\bar{x} \qquad \left[= -[k \;\; k_i]\begin{bmatrix}\hat{x}\\ x_i\end{bmatrix} \right]$$

STATE EQUATIONS FOR FOE COMPENSATOR

$$\underbrace{\frac{d}{dt}\begin{bmatrix}\hat{x}\\ x_i\end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\begin{bmatrix}A-LG & 0\\ 0 & 0\end{bmatrix}}_{\bar{A}}\underbrace{\begin{bmatrix}\hat{x}\\ x_i\end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix}B\\ 0\end{bmatrix}}_{\bar{B}_u}\underset{u}{u^*} + \underbrace{\begin{bmatrix}L\\ -1\end{bmatrix}}_{\bar{B}_y}\underset{y}{y} + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{\bar{B}_r}\underset{r}{r}$$

$$u^* = -\bar{k}\bar{x}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}_u u^* + \bar{B}_y y + \bar{B}_r r \quad (1)$$

$$u^* = -\bar{k}\bar{x} \quad (2)$$

(2) → (1) ⇒ $\dot{\bar{x}} = (\bar{A} - \bar{B}_u\bar{k})\bar{x} + \bar{B}_y y + \bar{B}_r r$

$$u^* = -\bar{k}\bar{x}$$

let $r = 0$.
⇒ only input is y.

⇒ SYSTEM IS $\left\{ \bar{A} - \bar{B}_u\bar{k},\ \bar{B}_y,\ -\bar{k},\ 0 \right\}$.

IN CLOSED LOOP $U = U^*$

$$y = \underbrace{\{A, B, C, 0\}}_{P} u$$

$$u^* = \underbrace{\{\bar{A} - \bar{B}_u \bar{k}, \bar{B}_y, -\bar{k}, 0\}}_{} y + \{\bar{A} - \bar{B}_u \bar{k}, \bar{B}_r, -\bar{k}, 0\} r$$

$$u^* = S1 y + S2 r$$

$$y = Pu$$

$$u = S1 y + S2 r.$$

$$\Rightarrow y = P(S1 y + S2 r)$$

$$(1 - P S1) y = P. S2 r$$

$$\frac{y}{r} = \frac{S2}{1 - P S1}.$$

FULL SYSTEM:

PLANT: $\dot{x} = Ax + Bu$

FOE: $\dot{\hat{x}} = (A-LC)\hat{x} + LCx + Bu$

INTEGRAL TERM: $\dot{x}_i = -Cx + r$

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \\ x_i \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ LC & A-LC & 0 \\ -C & 0 & 0 \end{bmatrix}}_{A_p}\begin{bmatrix} x \\ \hat{x} \\ x_i \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ B \\ 0 \end{bmatrix}}_{B_{up}} u + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_{ur}} r$$

OUTPUT EQN: $y = \underbrace{\begin{bmatrix} C & 0 & 0 \end{bmatrix}}_{C_p}\begin{bmatrix} x \\ \hat{x} \\ x_i \end{bmatrix}$

$$u = -\underbrace{\begin{bmatrix} 0 & k_{bar} \end{bmatrix}}_{k_p}\begin{bmatrix} x \\ \hat{x} \\ x_i \end{bmatrix}$$

SUBSTITUTE ABOVE

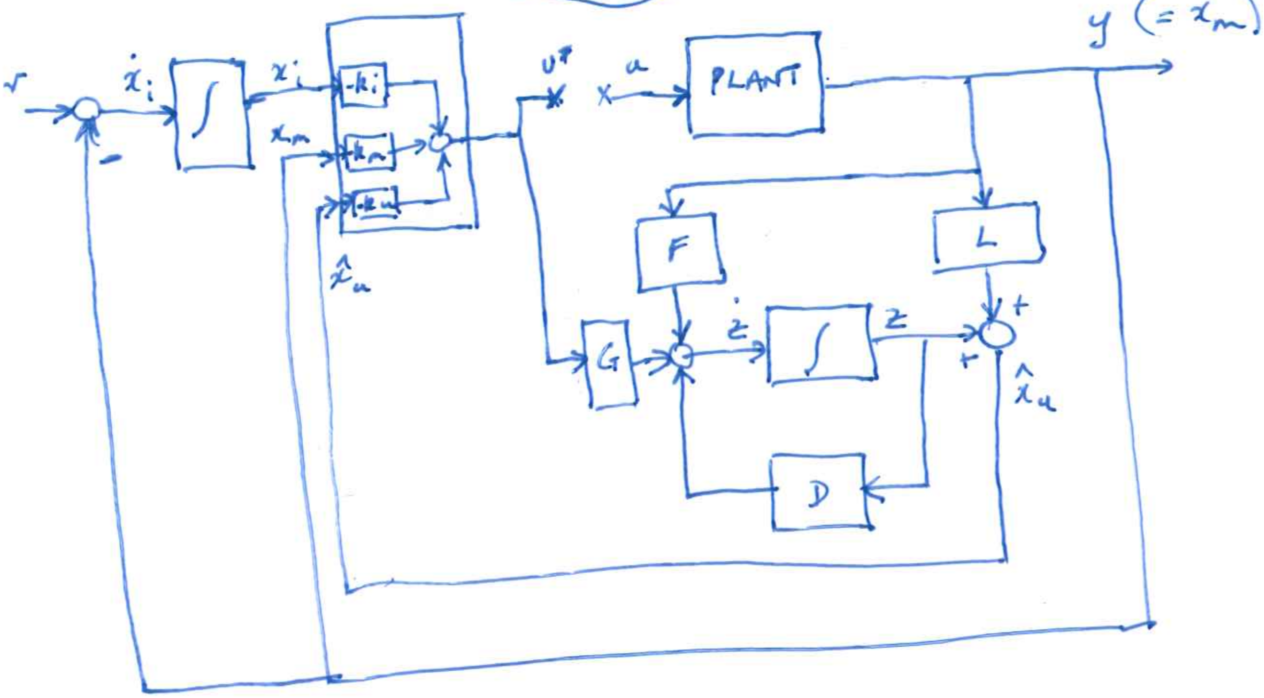
$\Rightarrow$

$$\dot{\bar{x}} = (A_p - B_{up}k_p)\bar{x} + B_{ur} r$$

$$y = C_p \bar{x}$$

$$-\begin{bmatrix} k_m & k_u & k_i \end{bmatrix}\begin{bmatrix} x_m \\ \hat{x}_u \\ x_i \end{bmatrix}$$

ROE



TRANFER FUNCTION OF THE COMPENSATOR (ROE)

$$\dot{z} = Dz + Gu + Fy$$

$$\hat{x}_u = z + Ly$$

$$\dot{x}_i = -y + r \qquad \bar{x} = \begin{bmatrix} x_m \\ \hat{x}_u \\ x_i \end{bmatrix}$$

$$u = -\bar{k}\bar{x}$$

$$u = -k_m y - k_u \hat{x}_u - k_i x_i \qquad \bar{k} = \begin{bmatrix} k_m & k_u & k_i \end{bmatrix}$$

$$\frac{d}{dt}\underbrace{\begin{bmatrix} z \\ \\ x_i \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} D & 0 \\ \\ 0 & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} z \\ \\ x_i \end{bmatrix} + \underbrace{\begin{bmatrix} G \\ \\ 0 \end{bmatrix}}_{\bar{B}_u} u + \underbrace{\begin{bmatrix} F \\ \\ -1 \end{bmatrix}}_{\bar{B}_y} y + \underbrace{\begin{bmatrix} 0 \\ \\ 1 \end{bmatrix}}_{\bar{B}_r} r$$

$$u = -k_m y - k_u(z + Ly) - k_i x_i$$

$$\Rightarrow \quad u = -k_u z - k_i x_i - (k_m + k_u L)y$$

$$\Rightarrow \quad u = -\underbrace{\begin{bmatrix} k_u & k_i \end{bmatrix}}_{\bar{k}_1} \begin{bmatrix} z \\ x_i \end{bmatrix} - (k_m + k_u L)y$$

$$\dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}_u u + \bar{B}_y\, y + \bar{B}_r\, r \qquad (1)$$

$$u = -\bar{k}_1 \bar{x} - (k_m + k_u L)\,y \qquad (2)$$

(2) → (1) →

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}_u\left(-\bar{k}_1\bar{x} - (k_m + k_u L)y\right) + \bar{B}_y y + \bar{B}_r r$$

$$\dot{\bar{x}} = \left(\bar{A} - \bar{B}_u k_1\right)\bar{x} + \left(\bar{B}_y - \bar{B}_u(k_m + k_u L)\right)y + \bar{B}_r r$$

$$\dot{\bar{x}} = \left(\bar{A} - \bar{B}_u \bar{k}_1\right)\bar{x} + \left[\bar{B}_y - \bar{B}_u\,(k_m + k_u L)\right]y + \bar{B}_r r$$

$$u = -\bar{k}_1\bar{x} - (k_m + k_u L)\,y.$$

$$\bar{x} = \begin{bmatrix} z \\ x_i \end{bmatrix}$$

$$\bar{k}_1 = [k_u \;\; k_i]$$

$$k_{bar} = [k_m \;\; k_u \;\; k_i]$$

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FULL SYSTEM

PLANT : $\dot{x} = Ax + Bu$

ROE : $$\frac{d}{dt}\begin{bmatrix} z \\ x_i \end{bmatrix} = \bar{A}\begin{bmatrix} z \\ x_i \end{bmatrix} + \bar{B}_u u + \bar{B}_y \overbrace{x_m}^{y} + \bar{B}_r r \qquad (1)$$

$$u = -\underbrace{\bar{k}_1}_{[k_u \;\; k_i]}\begin{bmatrix} z \\ x_i \end{bmatrix} - (k_m + k_u L) x_m \qquad (2)$$

(2) → (1) ⇒

$$\frac{d}{dt}\begin{bmatrix} z \\ x_i \end{bmatrix} = \bar{A}\begin{bmatrix} z \\ x_i \end{bmatrix} + \bar{B}_u\left(-\overbrace{\bar{k}_1}^{[k_u \;\; k_i]}\begin{bmatrix} z \\ x_i \end{bmatrix}\right) - \bar{B}_u(k_m + k_u L)x_m + \hat{B}_y x_m + \bar{B}_r r$$

$$\frac{d}{dt}\begin{bmatrix} z \\ x_i \end{bmatrix} = \left(\bar{A} - \bar{B}_u[k_u \;\; k_i]\right)\begin{bmatrix} z \\ x_i \end{bmatrix} + \left[\bar{B}_y - \bar{B}_u(k_m + k_u L)\right]x_m + \bar{B}_r r$$

$\bar{k}_1 = [k_u \;\; k_i]$

$$\frac{d}{dt}\begin{bmatrix} z \\ x_i \end{bmatrix} = \left(\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} - \underbrace{\begin{bmatrix} G \\ 0 \end{bmatrix}[k_u \;\; k_i]}_{\begin{bmatrix} Gk_u & Gk_i \\ 0 & 0 \end{bmatrix}}\right)\begin{bmatrix} z \\ x_i \end{bmatrix} + \left(\begin{bmatrix} F \\ -1 \end{bmatrix} - \begin{bmatrix} G \\ 0 \end{bmatrix}(k_m + k_u L)\right)\underbrace{x_m}_{Cx} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$\dot{z} = (D - Gk_u)z - Gk_i x_i + [F - G(k_m + k_u L)]\underbrace{x_m}_{Cx}$$

$$\dot{x}_i = -Cx + r$$

$$y = x_m = Cx.$$

PLANT:

$$\dot{x} = Ax + Bu \quad -(1)$$

$$u = -\begin{bmatrix} k_u & k_i \end{bmatrix}\begin{bmatrix} z \\ x_i \end{bmatrix} - (k_m + k_u L)Cx$$

$$u = -k_u z - k_i x_i - (k_m + k_u L)Cx. \quad -(2)$$

(2) → (1) $\Rightarrow \dot{x} = Ax + B\left(-k_u z - k_i x_i - (k_m + k_u L)\, Cx\right)$

$$\dot{x} = \left[A - B(k_m + k_u L)C\right]x - Bk_u z - Bk_i x_i$$

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$$\frac{d}{dt}\begin{bmatrix} x \\ z \\ x_i \end{bmatrix} = \begin{bmatrix} A - B(k_m + k_u L)C & -Bk_u & -Bk_i \\ (F - G(k_m + k_u L))C & D - Gk_u & -Gk_i \\ -C & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ z \\ x_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ z \\ x_i \end{bmatrix}$$

```
clear
close all

% % phase variable form
% A = [0   1
%      -9  -3]
% B = [0;  1]
% C = [9   0]

% dual phase variable form
A = [-3  1
     -9  0]
B = [0;  9]
C = [1   0]


sys_ss = ss(A,B,C,0)
sys_tf = tf(sys_ss)

co = ctrb(A,B)
rank(co)

p = [-10,-20]
K = acker(A,B,p)

sys_cl = ss(A-B*K,B,C,0)
figure(1)
step(sys_cl)
[y,t,~] = step(sys_cl);
stepinfo(y,t)

sys_loop = ss(A,B,K,0)
figure(2)
margin(sys_loop)

%------------------------------------

% Quiz 2

Abar = [A      zeros(size(A,1),1)
        -C     0                  ]

Bbar  = [B;    0]
Brbar = [zeros(size(A,1),1);    1]
Cbar  = [C     0]

R = 1;
Q = diag([0   0    10000])
kbar = lqr(Abar, Bbar, Q, R)

sys_fsfi = ss(Abar-Bbar*kbar, Brbar, Cbar, 0)
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figure(3)
step(sys_fsfi)

stepinfo(sys_fsfi)

eg = eig(Abar-Bbar*kbar)


sys_fsfi_loop = ss(Abar, Bbar, kbar, 0);
figure(4)
margin(sys_fsfi_loop)


%------------------------------------

% Quiz 3

% kbar is from Quiz 2
L = acker(A',C', 5*eg(1:2))'

Abar = [A-L*C                 zeros(size(A,1),1)
        zeros(1,size(A,2))    0                 ];

Bubar = [B;   0];
Bybar = [L;  -1];
Brbar = [0;   0;   1];
Cbar =  kbar;

sys_comp = ss(Abar-Bubar*kbar, Bybar, Cbar, 0);
sys_comp_zpk = zpk(sys_comp)

[sys,g] = balreal(sys_comp)    % compute balanced realization
elim = (g<1e-1);               % small entries of g -> negligible states
sys_comp_r = modred(sys,elim);

sys_comp_r_zpk = zpk(sys_comp_r)
g

sys_plant = ss(A,B,C,0);
sys_loop = sys_plant*sys_comp;
% sys_loop = sys_plant*sys_comp_r_zpk;
figure(5)
margin(sys_loop)

figure(6)
margin(sys_fsfi_loop)
hold on
margin(sys_loop)
hold off
```

```
%-----------------------------------------------
% transfer functions

P = ss(A,B,C,0);
S1 = ss(Abar-Bubar*kbar, Bybar, -kbar, 0);
S2 = ss(Abar-Bubar*kbar, Brbar, -kbar, 0);

sys = P*S2/(1-P*S1)
figure(7)
step(sys)

%-----------------------------------------------
% Step response using full system state equations

Ap = [A      zeros(size(A))        zeros(size(A,1),1)
      L*C    A-L*C                 zeros(size(A,1),1)
     -C      zeros(1,size(A,2))    0                   ]

Bup = [B;   B;   0]
Brp = [zeros(2*size(A,1),1); 1]

Cp = [C zeros(size(C)) 0]

kbar2 = [zeros(1,size(kbar,2)-1) kbar]
sys_full_FOE = ss(Ap-Bup*kbar2, Brp, Cp, 0)
figure(8)
step(sys_full_FOE)


%-----------------------------------------------
%  ROE design: Quiz 4
% to use the following code, need C to take the form:
%   C = [I  0]
% for our example using dual phase variable form we have
%   C = [1  0]

A11 = A(1,1);
A12 = A(1,2);
A21 = A(2,1);
A22 = A(2,2);

B1 = B(1);
B2 = B(2);

p = -1000;  % ROE pole location
L = acker(A22',A12',p)';
D = A22-L*A12;
F = D*L+A21-L*A11;
G = B2-L*B1;

Abar =  [D    0
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```
         0     0];
Bubar = [G;    0];
Bybar = [F;   -1];
Brbar = [0;    1];

% kbar = [km  ku  ki]
km = kbar(1);
ku = kbar(2);
ki = kbar(3);
k1 = kbar(2:3);


% transfer functions
P = ss(A,B,C,0);  % Plant
S1 = ss(Abar-Bubar*k1, Bybar-Bubar*(km+ku*L), -k1, -(km+ku*L)); % input: y
S2 = ss(Abar-Bubar*k1, Brbar, -k1, 0);  % input: r

sys = P*S2/(1-P*S1);
figure(9)
step(sys)

% compensator transfer function
sys_comp_zpk = zpk(S1)

% loop gain
sys_loop_ROE = -P*S1;
figure(10)
margin(sys_loop_ROE)

figure(11)
margin(sys_fsfi_loop)
hold on
margin(sys_loop_ROE)
hold off

%---------------------------------------------
% Full system equations

Abar = [A-B*(km+ku*L)*C       -B*ku          -B*ki
        (F-G*(km+ku*L))*C      D-G*ku         -G*ki
        -C                     0               0   ]

Brbar = [0;  0;   0;  1]
Cbar  = [C   0    0]

sys_full_ROE = ss(Abar,Brbar,Cbar, 0)
figure(12)
step(sys_full_ROE)
```