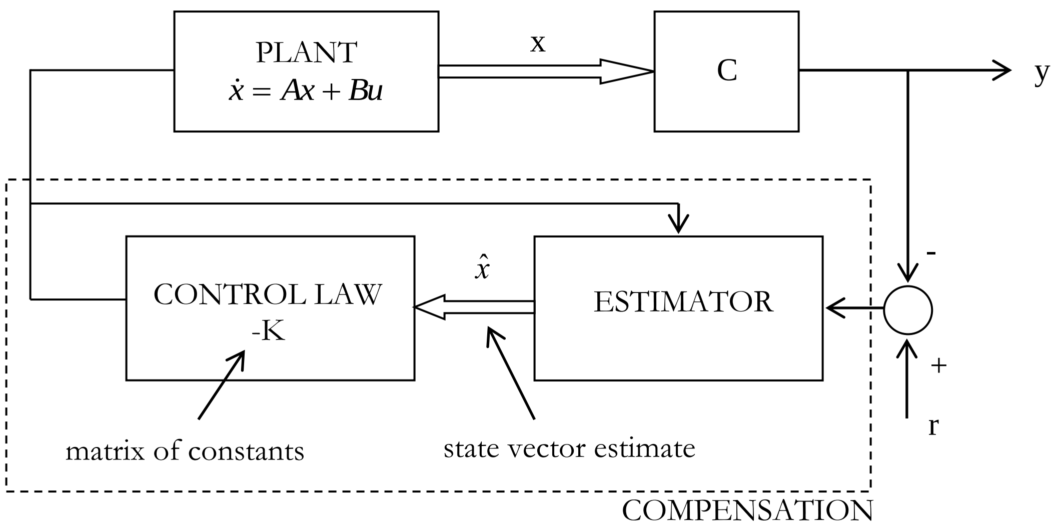
<u>Chapter 9</u>

Controller Design

Two Independent Steps:

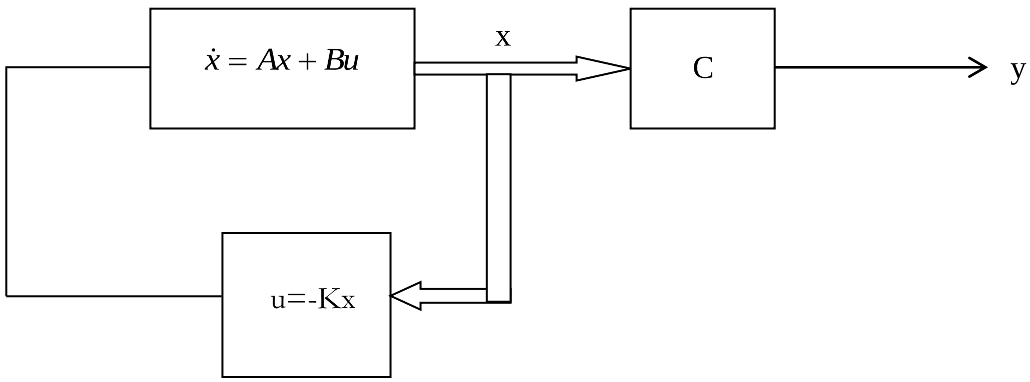
1) Feedback Design – Control Law **u=-Kx**
– assumes all states are accessible (a lot of sensors are necessary)

2) Design of Estimator – (also called an Observer) which estimates the entire state vector given the outputs and inputs



<u>Control Law Design</u>

Assumed system for control law design



u=-Kx
for an nth order system there are n feedback gains $K_1,\ldots,K_n$. By choice of K the roots can generally be placed anywhere

$$\dot{x} = Ax + Bu$$
$$u = -Kx$$
$$\dot{x} = (A - BK)x$$

characteristic equation is $|\lambda I - (A - BK)| = 0$

Placing Roots

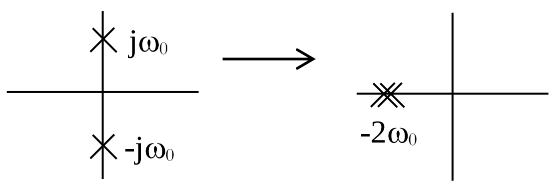
Example: Undamped oscillator with freq. ω_0

$$\frac{y(s)}{u(s)} = \frac{1}{s^2 + \omega_0^2}$$

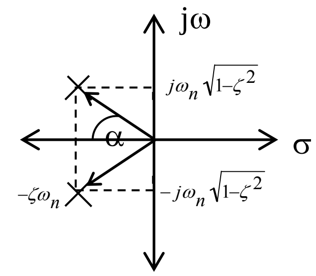
state space description

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

let's place the roots both at $-2\omega_0$



REVIEW



$$\cos\alpha = \zeta$$

→ we want to double the natural frequency and increase damping from $\zeta=0$ to $\zeta=1$

→ desired characteristic equation is

$$\Delta_d(s) = (s + 2\omega_0)^2 = s^2 + 4\omega_0 s + 4\omega_0$$

$$\det[sI - (A - BK)] = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left\{ \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \quad K_2] \right\} \right)$$

or $\quad s^2 + K_2 s + \omega_0^2 + K_1 = 0$

→ equating same coefficients: $\quad K_2 = 4\omega_0$, $\quad \omega_0^2 + K_1 = 4\omega_0^2$

→ $K_1 = 3\omega_0^2$, $K_2 = 4\omega_0$ → $K = [K_1 \quad K_2] = [3\omega_0^2 \quad 4\omega_0]$

Use of Canonical Forms

A simple way of calculating the gains when order is greater than three is to use special "canonical" forms of the state equations.

The special structure of the system matrix is referred to as companion form.

Example: Third order case:

The characteristic equation is

$$s^3 + a_1 s^2 + a_2 s + a_3$$

Recall the phase variable form (a lower companion form)

$$G(s) = \frac{b_1 s^{n-1} + ... + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + ... + a_n}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ & & & & : \\ & & & & 1 \\ -a_n & -a_{n-1} & .. & .. & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ : \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} b_n & b_{n-1} & ... & b_1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

The closed loop system matrix is

$$A - BK$$

Third order case:

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K_1 & K_2 & K_3 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 - K_1 & -a_2 - K_2 & -a_1 - K_3 \end{bmatrix}$$

Characteristic equation $|sI - (A - BK)| = 0$

$\rightarrow$ $$s^3 + (a_1 + K_3)s^2 + (a_2 + K_2)s + (a_3 + K_1) = 0 \qquad (1)$$

if the desired pole locations result in the characteristic equation

$$\Delta_d(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0 \qquad (2)$$

equating like coefficients of (1) and (2)

$$\rightarrow \quad \begin{aligned} K_3 &= -a_1 + \alpha_1 \\ K_2 &= -a_2 + \alpha_2 \\ K_1 &= -a_3 + \alpha_3 \end{aligned}$$

Example: Drill problem D9.1 page 635 of text

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$u = \begin{bmatrix} -k_1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + r$$

$$y = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find k_i to place the closed-loop system poles at

s= -3, -4, -5

ANS desired characteristic polynomial

$$\begin{aligned} \Delta_d(s) &= (s+3)(s+4)(s+5) \\ &= s^3 + 12s^2 + 47s + 60 \end{aligned}$$

$$\rightarrow \quad \alpha_1 = 12, \;\; \alpha_2 = 47, \;\; \alpha_3 = 60$$

we have

$$\begin{aligned} k_3 &= -a_1 + \alpha_1 \\ &= -7 + 12 = 5 \end{aligned}$$

$$\begin{aligned} k_2 &= -a_2 + \alpha_2 \\ &= -6 + 47 = 41 \end{aligned}$$

$$\begin{aligned} k_1 &= -a_3 + \alpha_3 \\ &= -3 + 60 = 57 \end{aligned}$$

control canonical (companion) form
i.e. phase variable form

Design Procedure

Given (A,B) and desired $\Delta_d(s)$, transform to (A_c, B_c) and solve for gains

We then need to transform gain back to original state space

note: the poles can only be placed arbitrarily if the system is fully controllable

This procedure is encapsulated in Ackermann's formula

Ackermann's Formula

$$k = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} M_C^{-1} \Delta_d(A)$$

where

$$M_C = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \text{ (controllability matrix)}$$

where n is the order of the system or the number of states and $\Delta_d(A)$ is defined as

$$\Delta_d(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \ldots + \alpha_n I$$

where the α_i's are the coefficients of the desired characteristic polynomial

$$\Delta_d(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$$

Example

Apply the formula for the undamped oscillator

$$\Delta_d(s) = (s + 2\omega_0)^2$$
$$= s^2 + 4\omega_0 s + 4\omega_0^{\ 2}$$

$\rightarrow \quad \alpha_1 = 4\omega_0, \qquad \alpha_2 = 4\omega_0^{\ 2}$

recall $\quad A = \begin{bmatrix} 0 & 1 \\ -\omega_0^{\ 2} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\rightarrow \quad \Delta_d(A) = \begin{bmatrix} -\omega_0^{\ 2} & 0 \\ 0 & -\omega_0^{\ 2} \end{bmatrix} + 4\omega_0 \begin{bmatrix} 0 & 1 \\ -\omega_0^{\ 2} & 0 \end{bmatrix} + 4\omega_0^{\ 2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3\omega_0^{\ 2} & 4\omega_0 \\ -4\omega_0^{\ 3} & 3\omega_0^{\ 2} \end{bmatrix}$$

also $M_c = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

→ $M_c^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

→ $K = [K_1 \quad K_2] = [0 \quad 1] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3\omega_0^2 & 4\omega_0 \\ -4\omega_0^3 & 3\omega_0^2 \end{bmatrix}$

→ $K = [3\omega_0^2 \quad 4\omega_0]$

which is the same as the result previously obtained.

Tracking Problems

For step input: Will find $\bar{N}$ to ensure zero steady-state error to step inputs

$$u = -Kx + \bar{N}r$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= (A - BK)x + B\bar{N}r \end{aligned}$$

now

$$\begin{aligned} \dot{x}_{ss} = 0 \Rightarrow 0 &= (A - BK)x_{ss} + B\bar{N}r \\ &\Rightarrow x_{ss} = -(A - BK)^{-1} B\bar{N}r \\ &\Rightarrow y_{ss} = -C(A - BK)^{-1} B\bar{N}r \end{aligned}$$

A-BK is stable → inverse exists

now

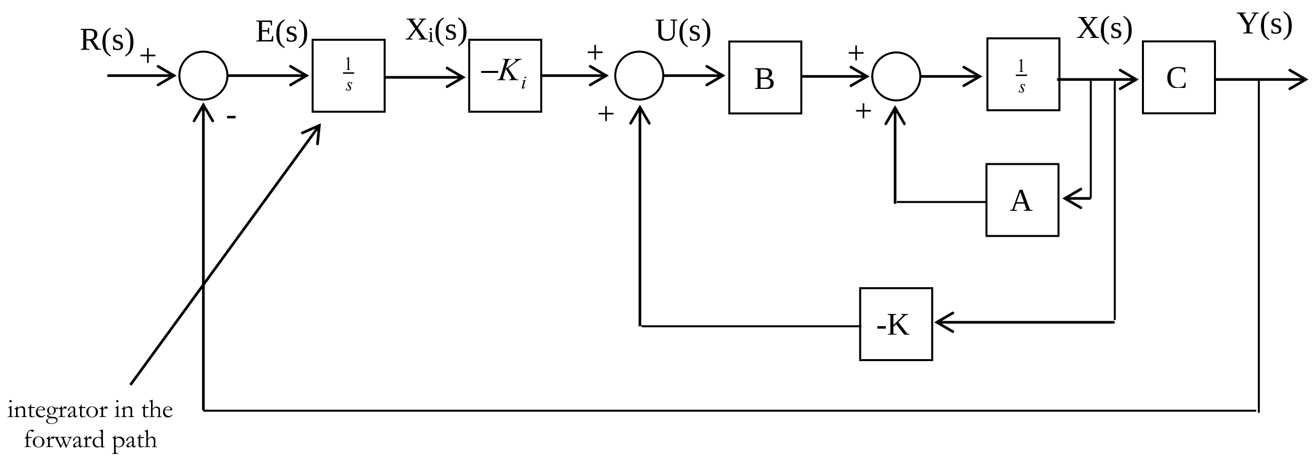
$$\begin{aligned} e_{ss} = r - y_{ss} &= r + C(A - BK)^{-1} B\bar{N}r \\ &= [1 + C(A - BK)^{-1} B\bar{N}]r \end{aligned}$$

steady-state error

→ to get zero steady state error

$$\bar{N} = \frac{-1}{C(A - BK)^{-1} B}$$

Integral Control (used to get zero steady state error)



Integrator increases the system order by one, i.e. augment the plant model by an added state variable x_i

$$x_i = \int e dt = \int (r - y) dt = \int (r - Cx) dt$$

$$\Rightarrow \dot{x}_i = r - Cx$$

Augmented systems becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

zero matrices (compatible dimensions)

Control law is $u = -Kx - K_i x_i = -\begin{bmatrix} K & K_i \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix}$

The design now proceeds as before.

Example

Double Integrator $\quad G(s) = \dfrac{1}{s^2}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Augment the plant

$$\begin{bmatrix} \dot{x} \\ \hline \dot{x}_i \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline -1 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \hline x_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \hline 0 \end{bmatrix} u$$

$$y = \left[\begin{array}{cc|c} 1 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \hline x_i \end{bmatrix}$$

Select poles of the closed loop system to be at

$$\{-1 \pm j, -5\}$$

N.B. 3 poles because of extra state

$$K = \begin{bmatrix} 12 & 7 & -10 \end{bmatrix}$$

Steady state output to a unit step input can be derived as follows

$$\dot{x} = Ax + Bu$$
$$\dot{x}_i = -Cx + r$$

$$\frac{d}{dt}\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix} + \overbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}^{B_u} u + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B_r} r$$

$$u = -\begin{bmatrix} K & K_i \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + B_u u + B_r r$$

$$u = -\bar{K}\bar{x}$$

$$y = Cx$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix}$$

$$y = \bar{C}\bar{x}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} - B_u\bar{K}\bar{x} + B_r r$$
$$\dot{\bar{x}} = (\bar{A} - B_u\bar{K})\bar{x} + B_r r$$

in steady state $\dot{\bar{x}}_{ss} = 0$

$$\implies \bar{x}_{ss} = -(\bar{A} - B_u\bar{K})^{-1}B_r r$$

$$\implies y_{ss} = -\bar{C}(\bar{A} - B_u\bar{K})^{-1}B_r r$$

For the example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

→ $$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad B_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad B_r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \bar{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$y_{ss} = -\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 12 & 7 & -10 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

→ only term of interest

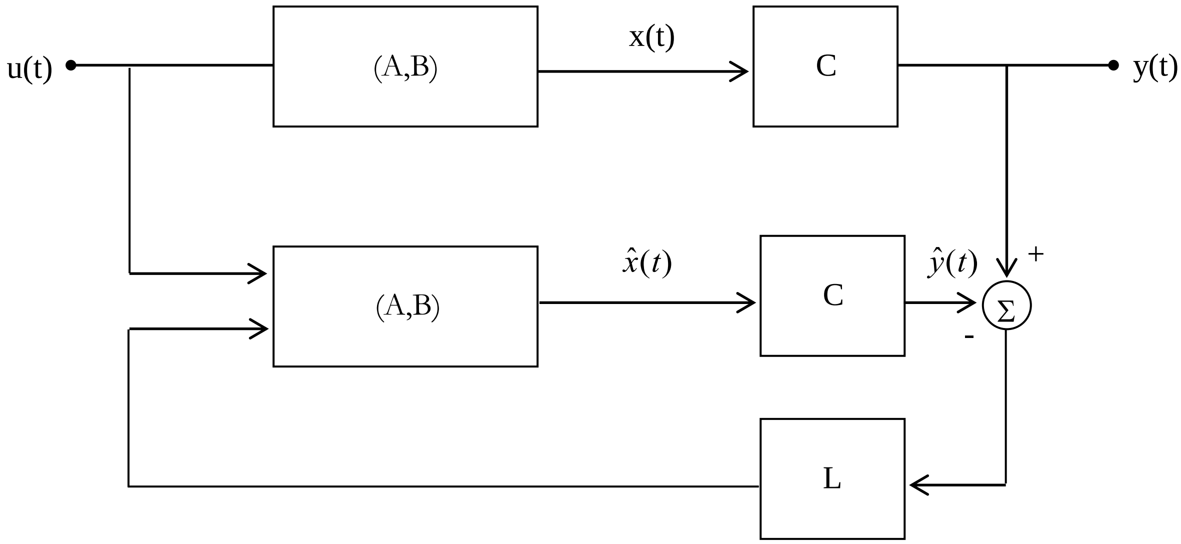
$$= -\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & \boxed{0} \\ -12 & -7 & 10 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$= 1 \cdot r$$

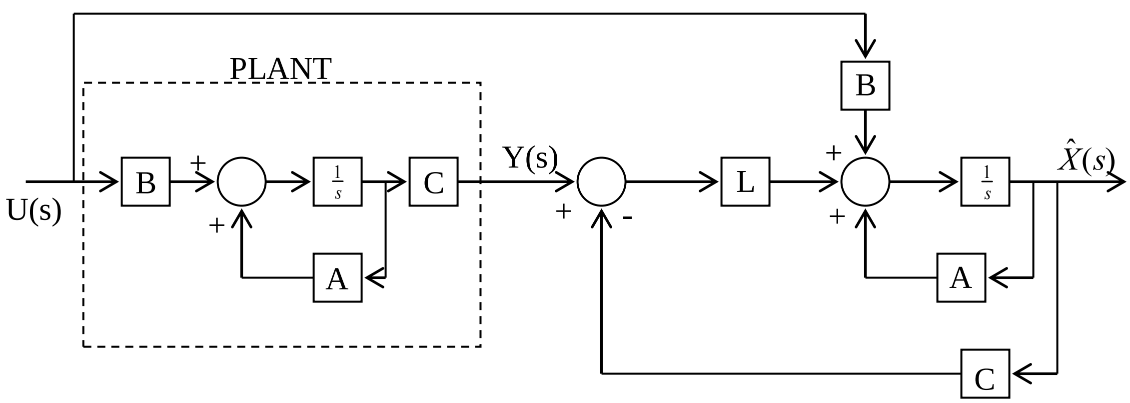
$$\det = -1 \begin{vmatrix} -12 & 10 \\ -1 & 0 \end{vmatrix} = -10$$

$$adj(A)_{13} = \begin{vmatrix} 1 & 0 \\ -7 & 10 \end{vmatrix} = 10$$

Observer Design



We will estimate states rather than measure them



Observer simulates the original system

<u>Original System</u>

$$\dot{x} = Ax + Bu \qquad x(0) = x_0$$
$$y = Cx$$

<u>Observer</u>

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad \hat{x}(0) = \hat{x}_0$$

Error between states and their estimates

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$$

$$\rightarrow \quad = Ax + Bu - A\hat{x} - Bu - L(y - C\hat{x})$$
$$= (A - LC)\tilde{x} \qquad \tilde{x}(0) = \tilde{x}_0$$

Observer error will go to zero asymptotically $\Leftrightarrow$ A-LC is stable (i.e. eigenvalues are in the LHP)

note: the eigenvalues of A-LC are the observer poles, which can be placed arbitrarily if the system is observable

this can be done by choice of the observer gains L (a column vector for single output systems)

Definition: 1) a system is <u>detectable</u> if the unstable modes are observable

2) a system is <u>stablizable</u> if the unstable modes are controllable

Duality

Control	Estimation	Control	Estimation
A	A^T	M_C	M_o^T
B	C^T	K	L^T
C	B^T		

<u>Example</u>

$$M_C = \begin{bmatrix} B & AB & \ldots. & A^{n-1}B \end{bmatrix}$$

duality

$$M_o^T = \begin{bmatrix} C^T & A^T C^T & \ldots. & A^{T^{n-1}} C^T \end{bmatrix}$$
$$= \begin{bmatrix} C^T & (CA)^T & \ldots. & (CA^{n-1})^T \end{bmatrix}$$
$$= \begin{bmatrix} C \\ CA \\ : \\ CA^{n-1} \end{bmatrix}^T \qquad \Rightarrow \qquad M_o = \begin{bmatrix} C \\ CA \\ : \\ CA^{n-1} \end{bmatrix}$$

Ackermann's formula to find observer gains

$$L = \Delta_e(A) M_o^{-1} \begin{bmatrix} 0 \\ 0 \\ : \\ 1 \end{bmatrix}$$

DERIVATION USING DUALITY

Ackermann's Formula for control problem

$$K = [0 \quad ..... \quad 1] M_C^{-1} \Delta_C(A)$$

Duality →

$$L^T = [0 \quad ..... \quad 1] M_o^{T^{-1}} \Delta_e(A^T)$$

$$= \left(\Delta_e(A) M_o^{-1} \begin{bmatrix} 0 \\ : \\ 1 \end{bmatrix} \right)^T$$

$$A^T B^T C^T = (CBA)^T$$

→ $$L = \Delta_e(A) M_o^{-1} \begin{bmatrix} 0 \\ : \\ 1 \end{bmatrix}$$

Example

Design an observer for

$$G(s) = \frac{1}{s^2}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] x$$

Note the system is completely observable

Design the observer with poles at $\{-2 \pm j2\}$

Actual characteristic equation:

$$|sI - (A - LC)| = \left| s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [1 \quad 0] \right|$$

$$= s^2 + l_1 s + l_2$$

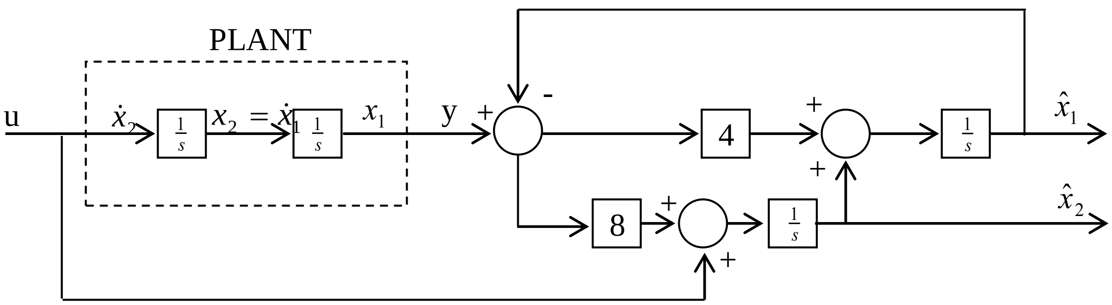
Desired characteristic equation:

$$(s+2+j2)(s+2-j2)=s^2+4s+8$$

Equating coefficients $\rightarrow$ $\begin{matrix} l_1 = 4 \\ l_2 = 8 \end{matrix}$ $\rightarrow$ $L=\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

The observer equations are

$$\dot{\hat{x}}_1 = \hat{x}_2 + 4(y-\hat{x}_1)$$
$$\dot{\hat{x}}_2 = 8(y-\hat{x}_1)+u$$



STRUCTURE OF OBSERVER

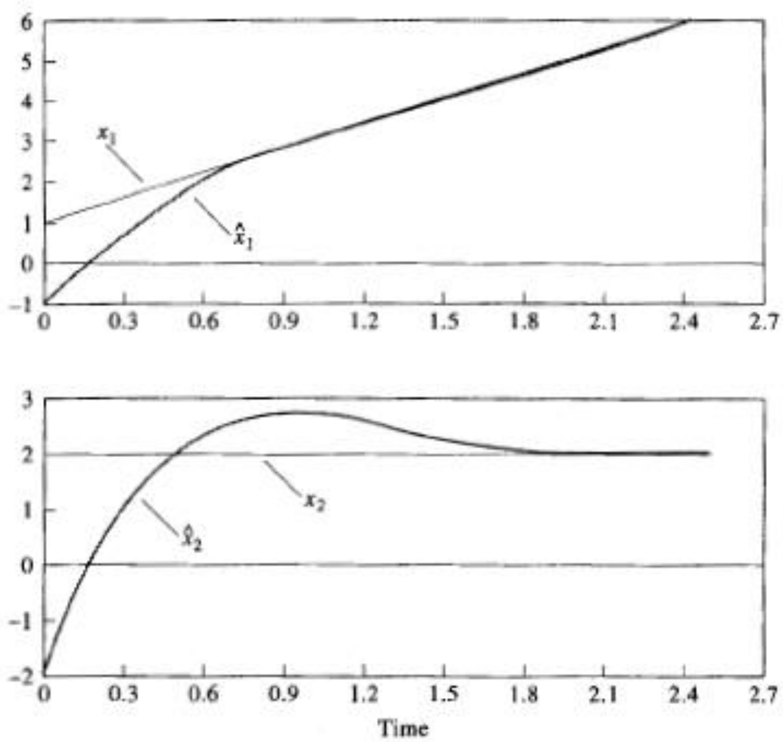


Figure 9.10 Simulation of the double-integrator plant and its observer. (a) Plot of the first state and its estimate. (b) Plot of the second state and its estimate.

Control Using Observers

Plant: $\dot{x} = Ax + Bu \qquad x(0) = x_0$

$y = Cx$

Observer: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad \hat{x}(0) = \hat{x}_0$

Estimated state feedback: $u = -K\hat{x}$

closing the loop

$$\dot{x} = Ax - BK\hat{x}$$

$$\begin{aligned}\dot{\hat{x}} &= (A - LC)\hat{x} - BK\hat{x} + Ly \\ &= (A - LC - BK)\hat{x} + LCx\end{aligned}$$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Separation Principle

Introduce transformation

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = P \begin{bmatrix} z \\ w \end{bmatrix} \qquad where \qquad P = \begin{bmatrix} I_N & 0_N \\ I_N & -I_N \end{bmatrix}$$

$$note \quad P^{-1} = P$$

$$\Rightarrow \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} I_N & 0_N \\ I_N & -I_N \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

Therefore using this transformation the old augmented state vector comprising x (the plant states) and $\hat{x}$ (the estimator sates) now becomes x and $\tilde{x}$ (the estimator error). The new system matrix $\tilde{A} = P^{-1}AP$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

note $\tilde{A}$ is block-triangular

Eigenvalues of a block-triangular matrix are equal to the eigenvalues of the diagonal blocks. So the eigenvalues of the full system comprise the eigenvalues of the plant (i.e. eigenvalues of A-BK) and the observer (i.e. eigenvalues of A-LC).

Alternative Approach

$$\dot{x} = Ax - BK\hat{x}$$
$$\dot{\hat{x}} = LCx + (A - BK - LC)\hat{x}$$

$$\begin{aligned} \dot{x} - \dot{\hat{x}} &= Ax - BK\hat{x} - LCx - (A - BK - LC)\hat{x} \\ &= (A - LC)x - (A - LC)\hat{x} \end{aligned}$$
$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$

$$\dot{x} = Ax - BK\hat{x}$$

now $\quad \tilde{x} = x - \hat{x} \quad \Rightarrow \quad \hat{x} = x - \tilde{x}$

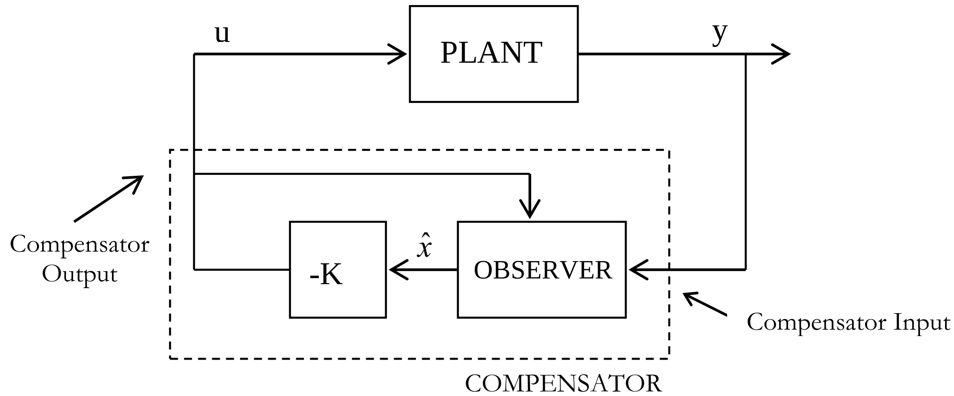
$$\begin{aligned} \Rightarrow \dot{x} &= Ax - BK(x - \tilde{x}) \\ &= (A - BK)x + BK\tilde{x} \end{aligned}$$

Compensator Transfer Function

$$H(s) = \frac{U(s)}{Y(s)}$$

Compensator output, which is the plant input

Compensator input, which is the plant output



we have

$$u = -K\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) = A\hat{x} - BK\hat{x} + Ly - LC\hat{x}$$

$$= (A - BK - LC)\hat{x} + Ly$$

$$\Rightarrow \hat{X}(s) = (sI - A + BK + LC)^{-1} LY(s)$$

$$\therefore U(s) = \underbrace{-K(sI - A + BK + LC)^{-1} L}_{H(s)} Y(s)$$

Design Issues

1) Problem with pole placement is that there is no control over compenstor poles and zeros

2) Optimum choice for observer initial conditions is

$$\hat{x}(0) = C'(CC')^{-1} y(0)$$

3) Choice of observer poles:

   i. Choose them to be faster than controller poles

   ii. Alternatively, choose them to be at plant zeros (if the sytem has RHP zeros, use their LHP images).

Reduced-Order Observer Design

If system has n states and m measurements, then an observer of order (n-m) will be sufficient.

If $y = Cx$ we will assume C has the structure:

$$C = \begin{bmatrix} I & 0 \end{bmatrix} \qquad I \in R^{mxm}, \quad 0 \in R^{mx(n-m)}$$

measured states

$$\therefore y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_m \\ x_u \end{bmatrix} = x_m$$

unmeasured states

$$\Rightarrow \begin{bmatrix} \dot{x}_m \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_m \\ x_u \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$\Rightarrow \dot{x}_u = A_{22} x_u + (\underbrace{A_{21} x_m + B_2 u}_{known\ input}) \quad \longleftarrow$$

dynamics of the unmeasured state

$$\dot{x}_m = \dot{y} = A_{11} y + A_{12} x_u + B_1 u \quad \longleftarrow$$

also $$\Rightarrow \underbrace{\dot{y} - A_{11} y - B_1 u}_{known\ measurement} = A_{12} x_u$$

output relationship

Summarizing:

$$\left.\begin{aligned} &\dot{x}_u = A_{22} x_u + (\underbrace{A_{21} x_m + B_2 u}_{known\ input}) && (1a) \\ &\underbrace{\dot{y} - A_{11} y - B_1 u}_{known\ measurement} = A_{12} x_u && (1b) \end{aligned}\right\} (1)$$

<u>Recall for full-order observer:</u>

Plant: $$\left.\begin{aligned} \dot{x} &= Ax + Bu \qquad x(0) = x_0 \\ y &= Cx \end{aligned}\right\} (2)$$

Observer: $$\left.\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad \hat{x}(0) = \hat{x}_0 \right\} (3)$$

comparing (1) and (2) shows the correspondence

$$\left.\begin{array}{l} x \leftarrow x_u \\ A \leftarrow A_{22} \\ Bu \leftarrow A_{21}x_m + B_2 u \\ y \leftarrow \dot{y} - A_{11}y - B_1 u \\ C \leftarrow A_{12} \end{array}\right\} \quad (4)$$

substituting (4) into (3) we get the reduced order equation

$$\left.\dot{\hat{x}}_u = A_{22}\hat{x}_u + A_{21}x_m + B_2 u + L(\underbrace{\overbrace{\dot{y} - A_{11}y - B_1 u}^{A_{12}x_u} - A_{12}\hat{x}_u}_{A_{12}\tilde{x}_u})\right\} \quad (5)$$

let us now define the estimator error

$$\tilde{x}_u = x_u - \hat{x}_u$$

therefore subtracting (5) from (1a) and using (1b) (i.e. $\dot{y} - A_{11}y - B_1 u = A_{12}x_u$)

we get

$$\left.\dot{\tilde{x}}_u = (A_{22} - LA_{12})\tilde{x}_u \right\} \quad (6) \qquad \longleftarrow$$

the error dynamics are given by this equation

Design proceeds by, given an A_{22} and A_{12} we choose an L to place estimator poles.

Rewriting (5) we have

$$\dot{\hat{x}}_u = (A_{22} - LA_{12})\hat{x}_u + (A_{21} - LA_{11})y + (B_2 - LB_1)u + L\dot{y} \qquad (7)$$

The presence of the derivative of the measurement (i.e. $\dot{y}$) is not good since this amplifies the noise. To get around this we introduce a new state z where

$$z = \hat{x}_u - Ly \qquad (8)$$

$$\Rightarrow \hat{x}_u = z + Ly \qquad (9)$$

substituting (9) into (7) leads to the final form of the reduced-order observer

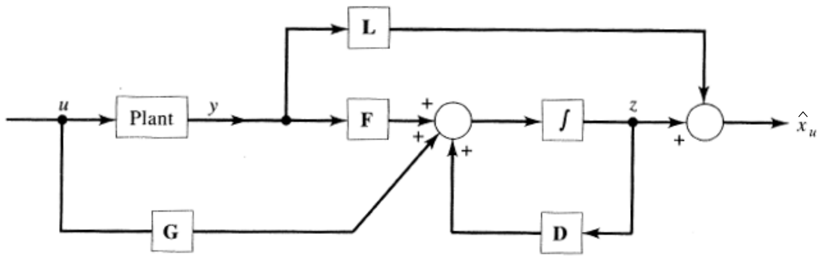
$$\begin{aligned} \dot{z} &= Dz + Fy + Gu \\ \hat{x}_u &= z + Ly \end{aligned}$$

z is the state of the estimator

where

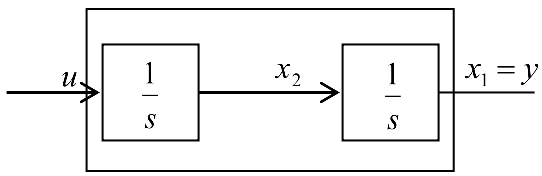
$$\begin{aligned} D &= A_{22} - LA_{12} \\ F &= DL + A_{21} - LA_{11} \\ G &= B_2 - LB_1 \end{aligned}$$

The block diagram of the reduced order observer is shown below



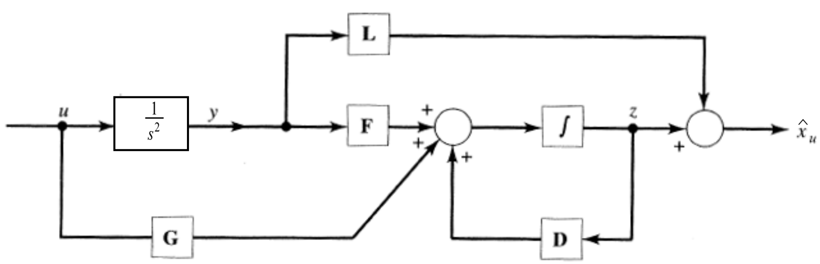
Example

Double integrator $G(s) = \frac{1}{s^2}$



$$\dot{x} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] x + \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] u \qquad \begin{array}{l} A_{11} = 0, \quad A_{12} = 1, \quad B_1 = 0 \\ A_{21} = 0, \quad A_{22} = 0, \quad B_2 = 1 \end{array}$$

$$y = \left[\begin{array}{c|c} 1 & 0 \end{array}\right] x$$



$$\dot{z} = Dz + Fy + Gu$$
$$\hat{x}_u = z + Ly$$

where

$$D = A_{22} - LA_{12}$$
$$F = DL + A_{21} - LA_{11}$$
$$G = B_2 - LB_1$$

Dynamics of reduced order observer $\dot{\tilde{x}}_u = (A_{22} - LA_{12})\tilde{x}_u$

Require observer pole at $s = -2$

$$\Rightarrow |\lambda I - A_{22} + LA_{12}| = 0 \qquad where \quad \lambda = -2$$

$$\Rightarrow \lambda I + L = 0$$

$$\Rightarrow L = 2$$

$$\Rightarrow D = -2$$

$$F = -4$$

$$G = 1$$

<u>Ackerman's formula for reduced order observer gains</u>

$$L = \Delta_e(A_{22}) M_0^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$M_0 = \begin{bmatrix} A_{12} \\ A_{12}A_{22} \\ A_{12}{A_{22}}^2 \\ \vdots \\ A_{12}{A_{22}}^{n-2} \end{bmatrix}$$

<u>for example</u>:

$$\Delta_e(s) = s + 2$$
$$\Delta_e(A_{22}) = 0 + 2$$

$$M_0 = A_{12} = 1$$

$$\therefore L = (2)(1)(1)$$
$$= 2$$

Reduced-Order Transfer Function

$$u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_m \\ \hat{x}_u \end{bmatrix} = -K_1 \overbrace{x_m}^{y} - K_2 \overbrace{\hat{x}_u}^{z+Ly}$$

now

$$\dot{z} = Dz + Fy + Gu$$
$$u = -K_1 y - K_2 (z + Ly)$$
$$u = -K_2 z - (K_1 + K_2 L) y$$

also

$$\dot{z} = Dz + Fy + G[-K_2 z - (K_1 + K_2 L) y]$$
$$= (D - GK_2) z + (F - GK_1 - GK_2 L) y$$

Transfer function: $\dfrac{U(s)}{Y(s)} = C'(sI - A')^{-1} B' + D'$

where

$$A' = D - GK_2$$

$$B' = F - GK_1 - GK_2 L$$

$$C' = -K_2$$

$$D' = -(K_1 + K_2 L)$$

When C is not of the form $\begin{bmatrix} I & 0 \end{bmatrix}$?

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$x = Qz \quad \Rightarrow \quad Q\dot{z} = AQz + Bu \quad \Rightarrow \quad \dot{z} = Q^{-1} AQz + Q^{-1} Bu$$

$$y = CQz + Du$$

so find a transformation Q so that CQ is of form $\begin{bmatrix} I & 0 \end{bmatrix}$

let $\quad Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$

so $\quad CQ = \begin{bmatrix} CQ_1 & CQ_2 \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}$

let $P = \begin{bmatrix} C \\ T \end{bmatrix}$ be nonsingular

arbitrary matrix

if $PQ = I$

$$PQ = \begin{bmatrix} C \\ T \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} CQ_1 & CQ_2 \\ TQ_1 & TQ_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$\therefore Q = P^{-1}$