Chapter 8

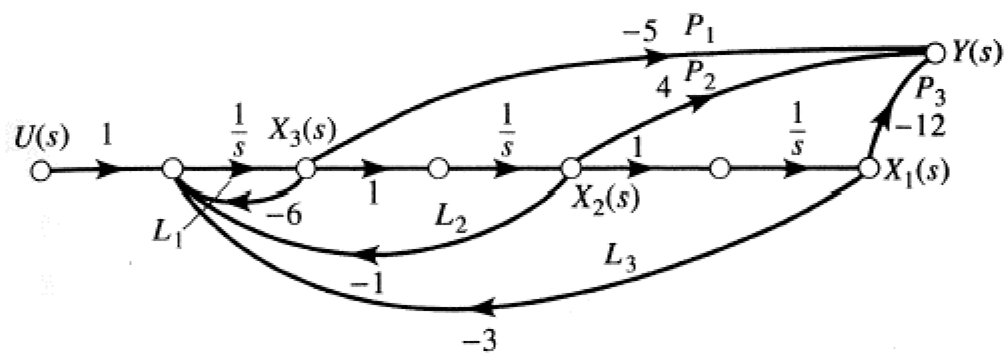
Transfer Function (TF) to State Space (SS) Form

Phase Variable Form **Mason's Rule** pg. 84-85

$$T(s) = \frac{-5s^2 + 4s - 12}{s^3 + 6s^2 + s + 3}$$

$$= \frac{-\frac{5}{s} + \frac{4}{s^2} - \frac{12}{s^3}}{1 + \frac{6}{S} + \frac{1}{S^2} + \frac{3}{S^3}}$$

$$= \frac{P_1 + P_2 + P_3}{1 - L_1 - L_2 - L_3}$$



The outputs of each integrator may be treated as state variables

$$X_1(s) = \tfrac{1}{s} X_2(s)$$
$$X_2(s) = \tfrac{1}{s} X_3(s)$$
$$X_3(s) = \tfrac{1}{s}[-3X_1(s) - X_2(s) - 6X_3(s) + U(s)]$$
$$Y(s) = -12X_1(s) + 4X_2(s) - 5X_3(s)$$

or

$$sX_1(s) = X_2(s)$$
$$sX_2(s) = X_3(s)$$
$$sX_3(s) = -3X_1(s) - X_2(s) - 6X_3(s) + U(s)$$
$$Y(s) = -12X_1(s) + 4X_2(s) - 5X_3(s)$$

→

$$\frac{dx_1}{dt} = x_2(t)$$
$$\frac{dx_2}{dt} = x_3(t)$$
$$\frac{dx_3}{dt} = -3x_1(t) - x_2(t) - 6x_3(t) + u(t)$$
$$y(t) = -12x_1(t) + 4x_2(t) - 5x_3(t)$$

let

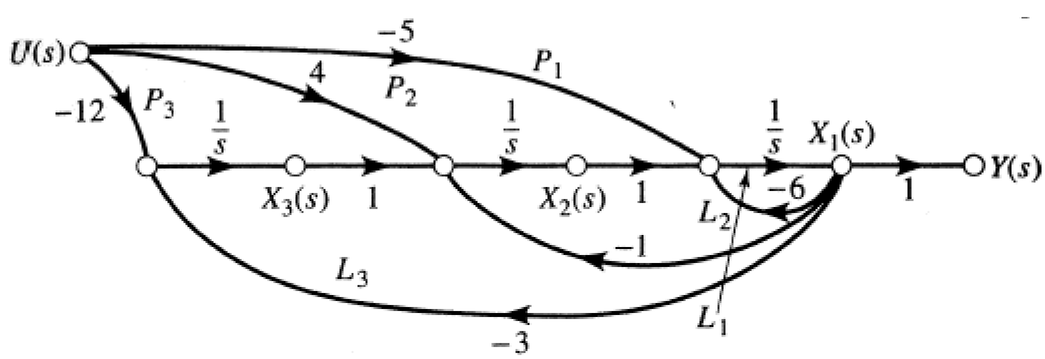
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \rightarrow \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

→

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -6 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{B} u$$

$$y = \overbrace{\begin{bmatrix} -12 & 4 & -5 \end{bmatrix}}^{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{0}^{D} u$$

<u>Alternatively</u> : <u>Dual Phase Variable Form</u>



$$sX_1(s) = -6X_1(s) + X_2(s) - 5U(s)$$
$$sX_2(s) = -X_1(s) + X_3(s) + 4U(s)$$
$$sX_3(s) = -3X_1(s) - 12U(s)$$
$$Y(s) = X_1(s)$$

$$\dot{x} = \begin{bmatrix} -6 & 1 & 0 \\ -1 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -5 \\ 4 \\ -12 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0u$$

In general

$$G(s) = \frac{b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

<u>Phase-Variable Form</u>

$$A = \begin{bmatrix} 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ : & . & & & : \\ : & & . & & 0 \\ : & & & . & 1 \\ -a_n & -a_{n-1} & .. & .. & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ : \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} b_n & b_{n-1} & .. & b_1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

<u>Dual Phase-Variable Form</u>

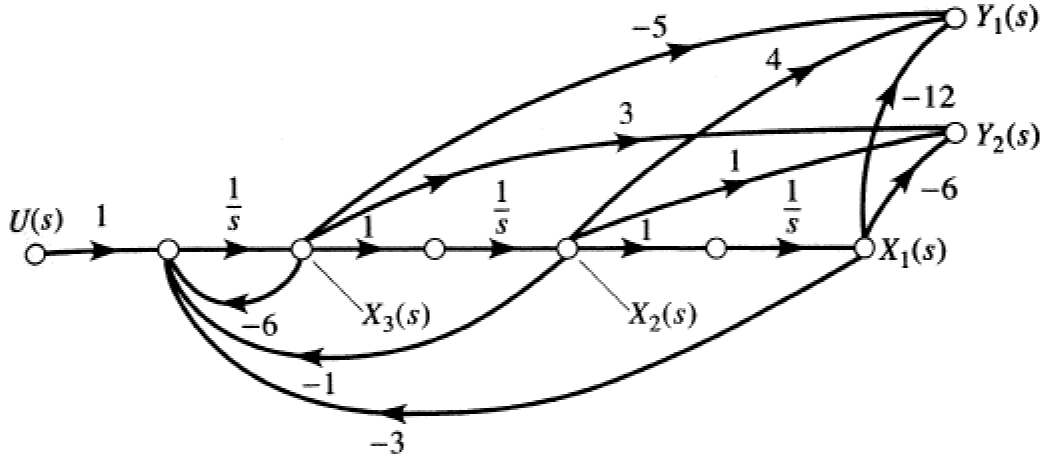
$$A = \begin{bmatrix} -a_1 & 1 & 0 & .. & 0 \\ -a_2 & 0 & 1 & .. & 0 \\ : & . & & & : \\ : & & . & & 0 \\ : & & & . & 1 \\ -a_n & 0 & .. & .. & 0 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & .. & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

<u>Multiple Inputs and Outputs</u>

Single input, two-output system

$$T_{11}(s) = \frac{-5s^2 + 4s - 12}{s^3 + 6s^2 + s + 3}$$

$$T_{21}(s) = \frac{3s^2 + s - 6}{s^3 + 6s^2 + s + 3}$$



$$Y(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} T_{11}(s) \\ T_{21}(s) \end{bmatrix}}_{T(s)} U(s)$$

$$T(s) = \frac{1}{s^3 + 6s^2 + s + 3} \begin{bmatrix} -5s^2 + 4s - 12 \\ 3s^2 + s - 6 \end{bmatrix}$$

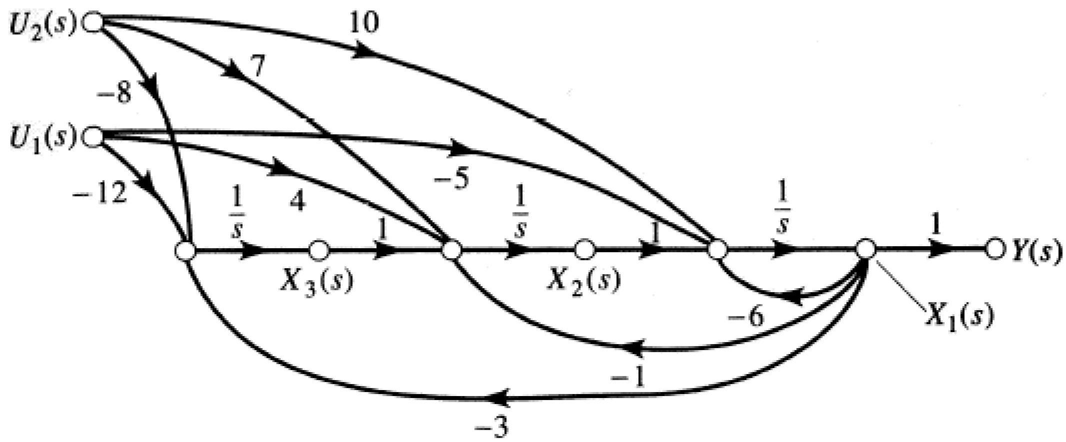
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -12 & 4 & -5 \\ -6 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

Two Inputs and One Output

$$T_{11}(s) = \frac{-5s^2 + 4s - 12}{s^3 + 6s^2 + s + 3}$$

$$T_{12}(s) = \frac{10s^2 + 7s - 8}{s^3 + 6s^2 + s + 3}$$



$$\dot{x} = \begin{bmatrix} -6 & 1 & 0 \\ -1 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -5 & 10 \\ 4 & 7 \\ -12 & -8 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \end{bmatrix} u$$

Transfer Function given State Space Form

$$\dot{x} = Ax + Bu \qquad (1)$$
$$y = Cx + Du \qquad (2)$$

$\pounds$ (1) → $sX - x(0) = AX + BU$

set x(0) = 0

→ $sX = AX + BU$

$(sI - A)X = BU$

$$X = \underbrace{(sI - A)^{-1}}_{\substack{resolvent \\ matrix}} BU$$

$$Y(s) = CX(s) + DU(s)$$

$\pounds$ (2) → $Y(s) = \underbrace{[C(sI - A)^{-1}B + D]}_{T(s) \leftarrow \text{ transfer function matrix}} U$

<u>Example: Two Input – Two Output</u>

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

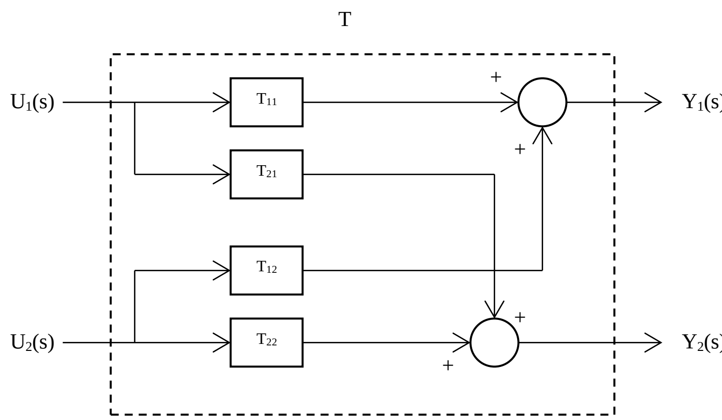
$$T(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}^{-1} \begin{bmatrix} 4 & 6 \\ -5 & 0 \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2} \begin{bmatrix} 1 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ -5 & 0 \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2} \begin{bmatrix} 1 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 4s-5 & 6s \\ -5s-23 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9s+18}{s^2+3s+2} & \frac{6s+12}{s^2+3s+2} \\ \frac{27s-63}{s^2+3s+2} & \frac{48s-12}{s^2+3s+2} \end{bmatrix} = \underbrace{\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix}}_{T}$$



$$Y = TU$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\rightarrow \quad \begin{aligned} Y_1 &= T_{11}U_1 + T_{12}U_2 \\ Y_2 &= T_{21}U_1 + T_{22}U_2 \end{aligned}$$

State Transformation

$$x(t) = Pz(t)$$

where:

x(t) is the old state
P is the non-singular matrix
z(t) is the new state

$$\rightarrow \quad \begin{aligned} \dot{x} &= P\dot{z} \\ \dot{x} &= Ax + Bu \end{aligned}$$

$$\rightarrow \quad P\dot{z} = APz + Bu$$

$$\rightarrow \quad \dot{z} = \underbrace{P^{-1}AP}_{\bar{A}}z + \underbrace{P^{-1}B}_{\bar{B}}u$$

$$y = Cx + Du$$

$$\rightarrow \qquad y = \underbrace{CP}_{\bar{C}}z + \underbrace{D}_{\bar{D}}u$$

New state space quadruple $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ for state z

where:

$$\bar{A} = P^{-1}AP$$
$$\bar{B} = P^{-1}B$$
$$\bar{C} = CP$$
$$\bar{D} = D$$

Change of state does not change the input-output relationship

$$\therefore T(s) = C(sI - A)^{-1}B + D \qquad \text{and} \qquad \bar{T}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

so that

$$T(s) = \bar{T}(s)$$

<u>Proof</u>

$$\begin{aligned}
\bar{T}(s) &= CP(sI - P^{-1}AP)^{-1}P^{-1}B + D \\
&= CP(sP^{-1}IP - P^{-1}AP)^{-1}P^{-1}B + D \\
&= CP[P^{-1}(sI - A)P]^{-1}P^{-1}B + D \\
&= CP \;\; P^{-1}(sI - A)^{-1}P\,P^{-1}B + D \qquad \text{using } (AB)^{-1} = B^{-1}A^{-1} \text{ if inverses exists} \\
&= C(sI - A)^{-1}B + D \\
&= T(s)
\end{aligned}$$

Diagonalization of System Matrix

<u>example</u>

$$\left.\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= x_1 + x_2 \\ y &= x_1 + x_2 \end{aligned}\right\} \quad \text{or} \quad \begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x \end{aligned}$$

change of state:

let $\quad x = Pz = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} z$

diagonal matrix

$$\bar{A} = P^{-1}AP$$

$$\bar{A} = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\overline{C} = CP = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix}$$

$$\rightarrow \quad \begin{aligned} \dot{z}_1 &= 0 \\ \dot{z}_2 &= 2z_2 \\ y &= 2z_2 \end{aligned}$$

solution is trivial

$$z_1(t) = z_1(0)$$
$$z_2(t) = e^{2t} z_2(0)$$

where $z(0) = P^{-1}x(0)$

What transformation matrix P diagonalizes the systems matrix?

Ans. $P = \begin{bmatrix} x_1 & : & x_2 & : & \ldots & x_n \end{bmatrix}$ where x_i are the eigenvectors of A if A has distinct eigenvalues

<u>example</u>

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

characteristic equation

<u>eigenvalues of A:</u>

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda + 2 & -1 \\ -1 & \lambda + 2 \end{vmatrix}$$

$$\Rightarrow (\lambda + 1)(\lambda + 3) = 0$$

eigenvalues $\left.\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= -3 \end{aligned}\right\} \textit{distinct} \Rightarrow \textit{diagonalizable}$

<u>eigenvectors of A:</u>

$$Ax_i = \lambda_i x_i$$

for $\lambda_1 = -1$

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = -1 \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$$

$$\Rightarrow -2x_{11} + x_{21} = -x_{11}$$
$$\Rightarrow \qquad x_{21} = x_{11}$$

choose $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda_2 = -3$

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = -3 \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$$

$$\Rightarrow x_{22} = -x_{12}$$

$$\text{let } x_{22} = 1 \Rightarrow x_{12} = -1$$

$$\text{therefore } x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

→ let $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

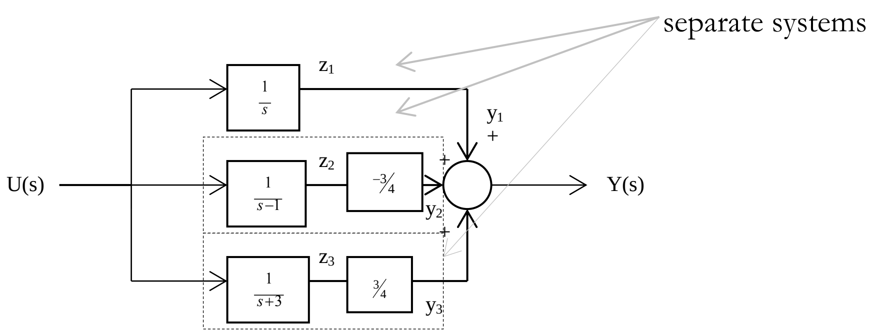
$$\begin{aligned}
\overline{A} &= P^{-1} A P \\
&= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \tfrac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -1 & -3 \end{bmatrix} \\
&= \tfrac{1}{2} \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
\end{aligned}$$

<u>Diagonalization Using Partial Fraction Expansion</u>

<u>Example</u>

$$T(s) = \frac{Y(s)}{U(s)} = \frac{s^2 - s - 3}{s(s-1)(s+3)} = \frac{1}{s} + \frac{-\frac{3}{4}}{s-1} + \frac{\frac{3}{4}}{s+3}$$

$$Y(s) = \frac{1}{s} U(s) + \frac{-\frac{3}{4}}{s-1} U(s) + \frac{\frac{3}{4}}{s+3} U(s)$$



$$z_1(s) = \frac{1}{s} U(s) \Rightarrow \dot{z}_1 = u \qquad y_1 = 1 z_1$$

$$z_2(s) = \frac{1}{s-1} U(s) \Rightarrow \dot{z}_2 = z_2 + u \qquad y_2 = -\frac{3}{4} z_2$$

$$z_3(s) = \frac{1}{s+3} U(s) \Rightarrow \dot{z}_3 = -3 z_3 + u \qquad y_3 = \frac{3}{4} z_3$$

$$y = y_1 + y_2 + y_3 = z_1 - \frac{3}{4} z_2 + \frac{3}{4} z_3$$

$$\rightarrow \quad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \quad ; \quad y = \begin{bmatrix} 1 & -\frac{3}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Complex Conjugate Characteristic Roots

Example

$$T(s) = \frac{6s^2 + 26s + 8}{(s+2)(s^2 + 2s + 10)} = \frac{-2}{s+2} + \frac{4+j}{s+1+j3} + \frac{4-j}{s+1-j3}$$

$$\rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1-j3 & 0 \\ 0 & 0 & -1+j3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & 4+j & 4-j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Inconvenient to use complex numbers, so don't expand into complex factors

$$\therefore T(s) = \frac{6s^2 + 26s + 8}{(s+2)(s^2 + 2s + 10)} = \frac{-2}{s+2} + \frac{8s + 14}{s^2 + 2s + 10}$$

$$\rightarrow \quad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & 14 & 8 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Example

$$T(s) = \frac{6}{s-3} + \frac{-8}{s+4} + \frac{-5s+1}{s^2 + 2s + 17} + \frac{8s}{s^2 - 3s + 10}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -17 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -10 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [6 \quad -8 \quad 1 \quad -5 \quad 0 \quad 8]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

<u>Example</u>

$$T(s) = \frac{10s^2 + 51s + 56}{(s+4)(s+2)^2} = \frac{3}{s+4} + \frac{7}{s+2} + \frac{-3}{(s+2)^2}$$

JORDAN CANONICAL FORM

$$\rightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [3 \quad -3 \quad 7]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

JORDAN BLOCK

Time Response of Systems

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$= e^{A(t-t_0)}x(t_0) + A^{-1}[e^{A(t-t_0)} - I]BU$$ where **u(t)=U** is a constant

where $e^{At} = \pounds^{-1}\{[sI - A]^{-1}\}$

Find e^{At} when $A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$

$$e^{At} = \text{£}^{-1}\{[sI - A]^{-1}\}$$
$$= \text{£}^{-1}\{\begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}^{-1}\}$$
$$= \text{£}^{-1}\begin{bmatrix} \frac{s}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s+3}{s^2+3s+2} \end{bmatrix}$$
$$= \text{£}^{-1}\begin{bmatrix} \frac{-1}{s+1} + \frac{2}{s+2} & \frac{1}{s+1} + \frac{-1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{2}{s+1} + \frac{-1}{s+2} \end{bmatrix}$$
$$= \begin{bmatrix} -e^{-t} + 2e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

Stability

$$\dot{x} = Ax \qquad\qquad x(0) = x_0$$

System is asymptotically stable if all states approach zero with time - i.e. **x(t)→0 as t→**∞

This will happen if the eigenvalues have negative real parts.

Bounded-input, bounded-output (BIBO) stability means that the system output is bounded for all bounded inputs. That is

$$|u(t)| \le N < \infty \quad \Rightarrow \quad |y(t)| \le M < \infty$$

In the absence of pole-zero cancellations, transfer function poles are identical to system eigenvalues, hence BIBO stability and asymptotic stability are equivalent.

Example

$$T(s) = \frac{s-1}{(s-1)(s+2)}$$

$$\left(\begin{array}{l} \text{Final Value Theorem:} \\ \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \end{array} \right)$$

pole at s=-2

no pole at s=+1 since: $\lim_{s\to 1} T(s) = \lim_{s\to 1} \frac{s-1}{(s-1)(s+2)} = \lim_{s\to 1} \frac{1}{2s+1} = \frac{1}{3} \neq \infty$

since only pole is at s=-2

→ system is BIBO stable since no pole in RHP or on imaginary axis

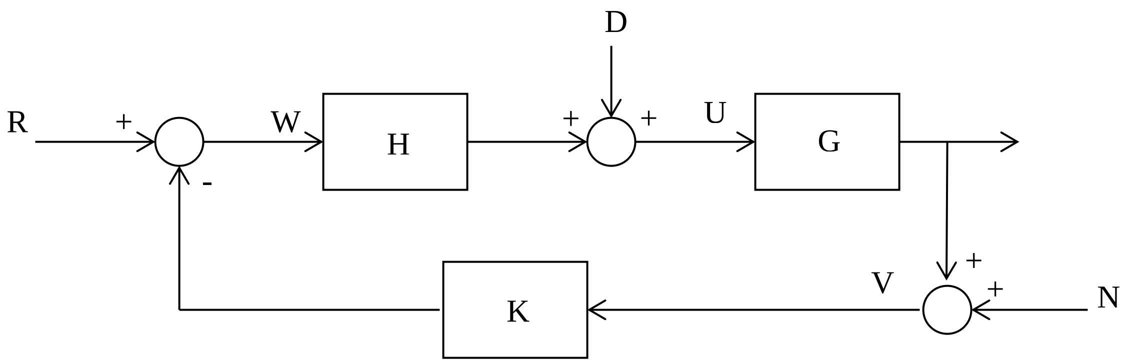
State-space realization

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$$

Eigenvalues are 1 and -2 → system is unstable in the asymptotic sense

Internal Stability

Based on transfer function description and is stronger than BIBO stability



General feedback system with disturbances

Internal stability requires that all signals within the feedback system remain bounded for all bounded inputs

Requires nine transfer functions from inputs R,D,N to outputs U,V,W be stable

Sufficient if
1) 1+KGH has no zeros in RHP and on imaginary axis
2) KGH has no pole-zero cancellations in RHP or on imaginary axis

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Controllability

A system is completely controllable if the system state $x(t_f)$ at time t_f can be forced to take on any desired value by applying a control input u(t) over a period of time from t_o until t_f.

Observability

A system is completely observable if any initial state vector $x(t_0)$ can be reconstructed by examining the system output y(t) over some period of time from t_0 to t_f.

Tests for Controllability and Observability

If the system matrix is diagonal then the tests are easy

e.g.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} x$$

eigenvalues are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$
modes are $e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}, e^{\lambda_4 t}$

Modes $e^{\lambda_3 t}$ *and* $e^{\lambda_4 t}$ *are uncontrollable since they are not connected to the control input*

Modes $e^{\lambda_2 t}$ *and* $e^{\lambda_4 t}$ *are unobservable since they are not connected to the output*

→ mode $e^{\lambda_1 t}$ controllable and observable
mode $e^{\lambda_2 t}$ controllable but unobservable
mode $e^{\lambda_3 t}$ uncontrollable but observable
mode $e^{\lambda_4 t}$ uncontrollable and unobservable

For MIMO systems
-Uncontrollable modes correspond to zero rows of B
-Unobservable modes correspond to zero columns of C

Controllability Matrix

$$\dot{x} = Ax + Bu$$

$$M_C = \begin{bmatrix} B & \vdots & AB & \vdots & .... & \vdots & A^{n-1}B \end{bmatrix}$$

A system is completely controllable if and only if M_c has full rank

Example

$$A = \begin{bmatrix} -4 & 1 \\ 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$M_C = \begin{bmatrix} B & \vdots & AB \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

det(M_c)=0 → not fully state controllable

Observability Matrix

$$M_O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

A system is completely observable if and only if M_O has full rank

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$M_O = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$$

M_O has two linearly independent rows (1 and 3) → fully observable

$$\left| M_O{}^T M_O \right| = \left| \begin{bmatrix} 1 & -2 & 0 & 0 \\ -1 & 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 0 & -1 \\ 0 & 2 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 5 & -5 \\ -5 & 10 \end{vmatrix}$$

$$= 25 \neq 0$$

→ fully observable