Chapter 10

Linear Quadratic Regulator Problem

Minimize the cost function J given by

$$J = \tfrac{1}{2}\int_0^\infty (x'Qx + u'Ru)dt$$

R > 0 positive definite (symmetric with positive eigenvalues)
Q ≥ 0 positive semi definite (symmetric with nonnegative eigenvalues)

subject to

$$\dot{x} = Ax + Bu$$
$$(y = Cx)$$

LQR SOLUTION:
Find the positive-definite solution P of the ARE (Algebraic Ricatti Equation)

$$A'P + PA + Q - PBR^{-1}B'P = 0$$
$$u = -Kx \quad \text{where} \quad K = R^{-1}B'P$$

The positive-definite solution of the ARE results in an asymptotically stable closed-loop system if:

1) the system is controllable
2) R > 0
3) $Q = C_q{}'C_q$ where (C_q,A) is observable

These conditions are necessary and sufficient

We can define another output z where

$z = C_q x$ → controlled or regulated output

Therefore $x'Qx = x'C_q{}'C_q x = z'z$

LQR design of double integrator

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_q = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

assume $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = 1$

$$Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = C_q{}'C_q$$

(A, B) is controllable
(C_q, A) is observable

note P is symmetric

ARE:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \underbrace{\underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & p_2 \\ 0 & p_3 \end{bmatrix}}\underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}}_{\begin{bmatrix} p_2^2 & p_2 p_3 \\ p_2 p_3 & p_3^2 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

solving

$$\begin{bmatrix} 0 & 0 \\ p_1 & p_2 \end{bmatrix} + \begin{bmatrix} 0 & p_1 \\ 0 & p_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_2^2 & p_2 p_3 \\ p_2 p_3 & p_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$1 - p_2^2 = 0 \;\Rightarrow\; p_2^2 = 1$ (1)

$p_1 - p_2 p_3 = 0$ (2)

$p_1 - p_2 p_3 = 0$ (3)

$\Big\}\; p_1 = p_2 p_3$

$2p_2 - p_3^2 = 0$ (4)

$$\Rightarrow p_2 = 1$$
$$p_3 = \sqrt{2}$$
$$p_1 = \sqrt{2}$$

$$\Rightarrow \quad P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$$

$$K = R^{-1}B'P = 1[0 \quad 1]\begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} = [1 \quad \sqrt{2}]$$

The closed loop system matrix becomes

$$A - BK = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}$$

Closed loop roots are:

$$\lambda^2 + \sqrt{2}\lambda + 1 = 0$$
$$\lambda = \frac{\sqrt{2}}{2}(-1 \pm j)$$

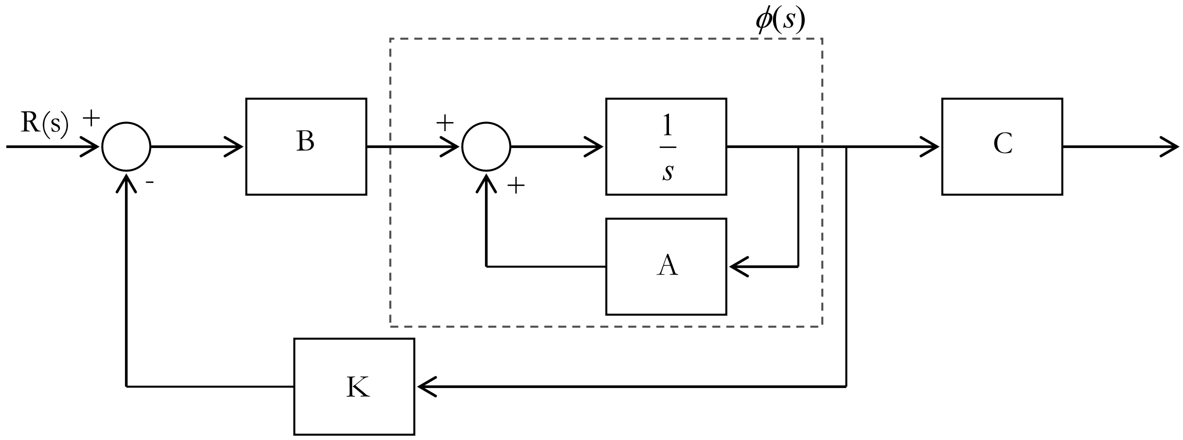
damping ratio is 0.707

The loop transfer function is:

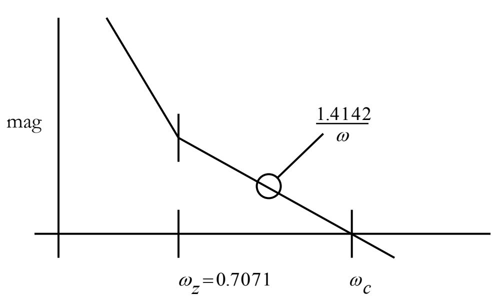
$$
\begin{aligned}
&K\phi(s)B \\
&= K(sI - A)^{-1}B \\
&= \frac{\sqrt{2}\left[s + \frac{\sqrt{2}}{2}\right]}{s^2}
\end{aligned}
$$

→ 65° phase margin

→ infinite gain margin



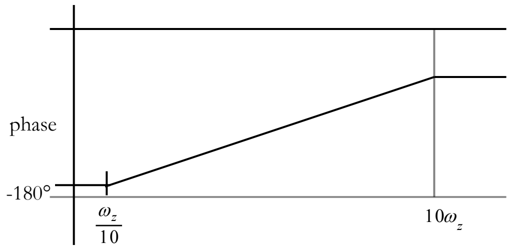
$$
\begin{aligned}
L(s) &= \frac{\sqrt{2}\left[s + \frac{\sqrt{2}}{2}\right]}{s^2} \\
&= \frac{\sqrt{2}\,\frac{\sqrt{2}}{2}\left[1 + \frac{2}{\sqrt{2}}s\right]}{s^2} \\
&= \frac{1 + \frac{s}{\frac{1}{\sqrt{2}}}}{s^2} \\
&= \frac{1 + \frac{s}{0.7071}}{s^2}
\end{aligned}
$$



$$
\left|L(j\omega)\right|_{\omega > 0.7071} \approx \left|\frac{\frac{j\omega}{0.7071}}{(j\omega)^2}\right| = \frac{1.4142}{\omega}
$$

$$|L(j\omega_c)| = 1 = \frac{1.4142}{\omega_c}$$

$$\omega_c = 1.4142$$



Phase@ ω_c = $-180 + \underbrace{\tan^{-1}\left(\frac{\omega_c}{0.7071}\right)}_{\text{phase margin}}$

Phase margin = $\tan^{-1}\frac{1.4142}{0.7071} = \tan^{-1}(2) = 63.4^\circ$

USING MATLAB TO GET EXACT RESULTS

<u>Matlab</u>

```
num = sqrt(2)*[ 1       sqrt(2)/2 ]

den = [ 1       0       0 ]
```

results: margin(num,den)
gm=∞
pm=65.53 @ ω=1.554

<u>Properties of LQR design</u>

From the ARE we can derive the relation

$$|1 + L(j\omega)|^2 = 1 + \tfrac{1}{\rho}|G_q(j\omega)|^2 \qquad (*)$$

ρ is a scalar

where $L(s) = K\phi(s)B$ -loop gain

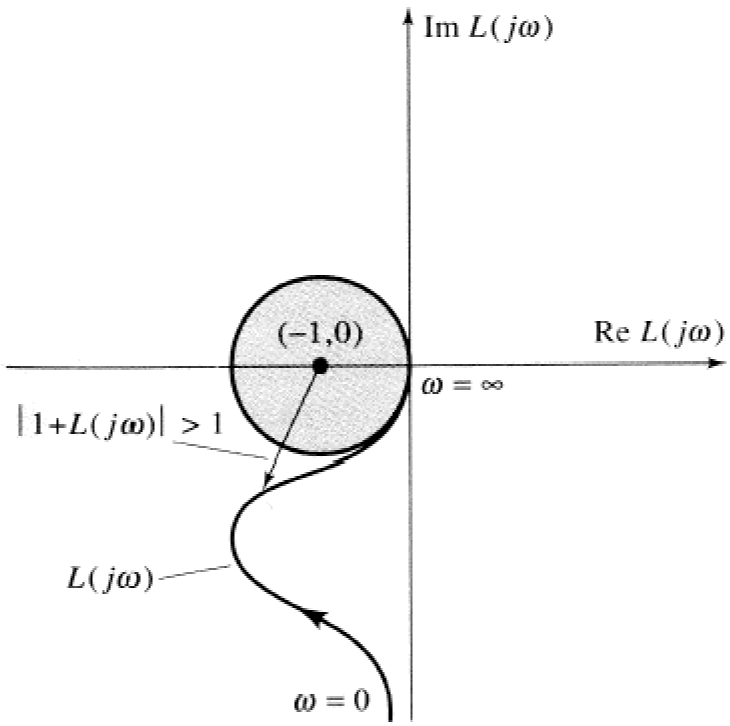
$\phi(s) \equiv (sI - A)^{-1}$

and
$$Q = C_q{}'C_q$$
$$G_q(s) = C_q\phi(s)B$$

From (*) we see

$$|1 + L(j\omega)| \geq 1$$

This implies that the Nyquist plot of the loop transfer function of an LQR design always stays outside of a unit circle centered at (-1,0).



In SISO case, LQR design has > 60° phase margin, infinite gain margin and a gain reduction tolerance of -6dB (i.e. the gain can be reduced by a factor of $\frac{1}{2}$ before instability occurs).

Recall pole placement does not guarantee stability margins.

<u>High-frequency roll-off rate</u>

Closed loop transfer function $T(j\omega) = -K(j\omega I - A + BK)^{-1} B$

$$\lim_{\omega\to\infty} T(j\omega) = \tfrac{-1}{j\omega} KB = \tfrac{-1}{j\omega} R^{-1} B' PB < 0$$

→ -20dB/dec roll off rate at high frequencies
- not good for noise suppression

Optimal Observers – Kalman Filter

State estimation – plant represented as

$$\dot{x} = Ax + Bu + \omega \quad \leftarrow \textit{process noise}$$
$$y = Cx + v \quad \leftarrow \textit{measurement noise}$$

The optimal filter is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

where $L = \Sigma C' R_0^{-1}$

where Σ is the positive definite solution of

$$A\Sigma + \Sigma A' + Q_0 - \Sigma C' R_0^{-1} C\Sigma = 0$$

Q_o and R_o are noise covariance matrices, which represent the intensity of the process and sensor noise inputs.

Require $Q_0 \geq 0, R_0 > 0$ and system to be observable.

If we combine the Kalman-Bucy Filter (optimal estimator) with LQR design, we have LQG (Linear Quadratic Gaussian). Let's do a LQG design for double integrator plant. We already have the LQR design.

For Kalman filter, assume

$$Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } R_0 = 1$$

Solving Ricatti equation with $\Sigma = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

we find
$$a^2 = 2b + 1$$
$$ab = c$$
$$b^2 = 1$$

→
$$\Sigma = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

and $L = \Sigma C' R_0^{-1} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

Transfer function of compensator is given by

$$H(s) = K\left(sI - A + BK + LC\right)^{-1} L$$

$$= \frac{3.14\,(s + 0.3)}{(s + 1.57 + j1.4)\,(s + 1.57 - j1.4)}$$

<u>Comparison of LQR and LQG</u>

-LQR has guaranteed stability margins
-LQG has no guaranteed stability margins

-high freq. roll off in LQG can be > 20 dB/dec exhibited by LQR → greater noise filtering in LQG

-LQG is not robust → uncertainty in plant may cause system to go unstable

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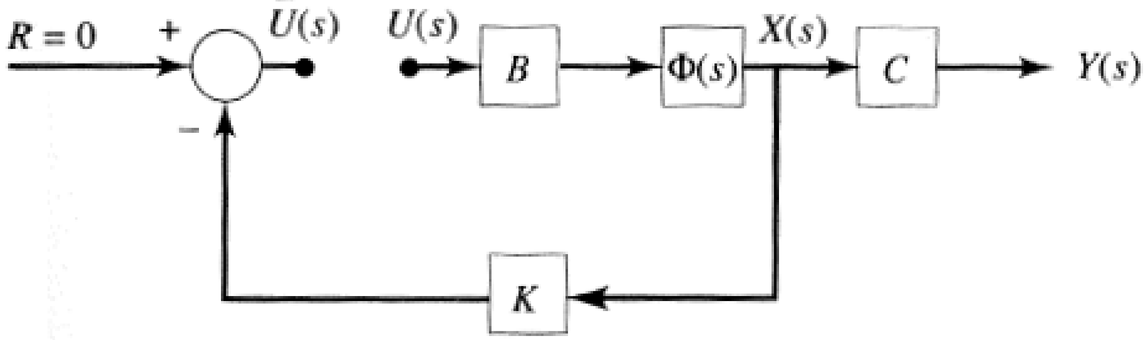
Loop Transfer Recovery (LTR)

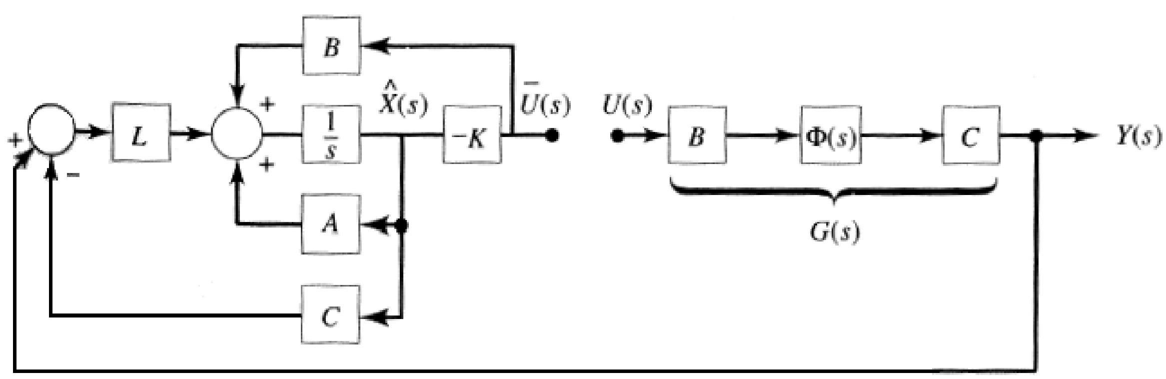
LQR → > 60° phase margin
infinite gain margin

LQG → no guaranteed margins

The properties of LQR can be recovered asymptotically by using Q_o and R_0 as tuning parameters



LQR loop gain, $L(s) = K\phi(s)B$
$= K(sI - A)^{-1}B$



LQG → $s\hat{X} = (-BK + A - LC)\hat{X} + LY(s)$

$$\frac{\hat{X}(s)}{Y(s)} = (sI - A + BK + LC)^{-1}L$$

loop gain, $L_{LQG}(s) = K(sI - A + BK + LC)^{-1}LC\phi(s)B$

If the following two conditions hold then LQR loop properties can be recovered if

1) G(s) is minimum phase
2) $R_0=1$ and $Q_0=q^2BB'$

Then it can be shown

$$\lim_{q\to\infty} L_{LQG}(s) = L(s)$$

The variable y that is recovered may be different from the variable z that is to be controlled

where $y = Cx$ and $z = C_q x$

<u>Loop Shaping Steps</u>

1) Determine the controlled variable and set

   $Q=C'C$ and $Q=C_q'C_q$

2) Get a desired loop gain in LQR design. Use R as tuning parameter.

3) Select scalar q and solve the filter Ricatti equation

$$A\Sigma + \Sigma A' + q^2 BB' - \Sigma C'C\Sigma = 0$$
$$L = \Sigma C'$$

4) Increase q until the resulting loop transfer function is close to the LQR design

<u>Do not make q too high since</u>

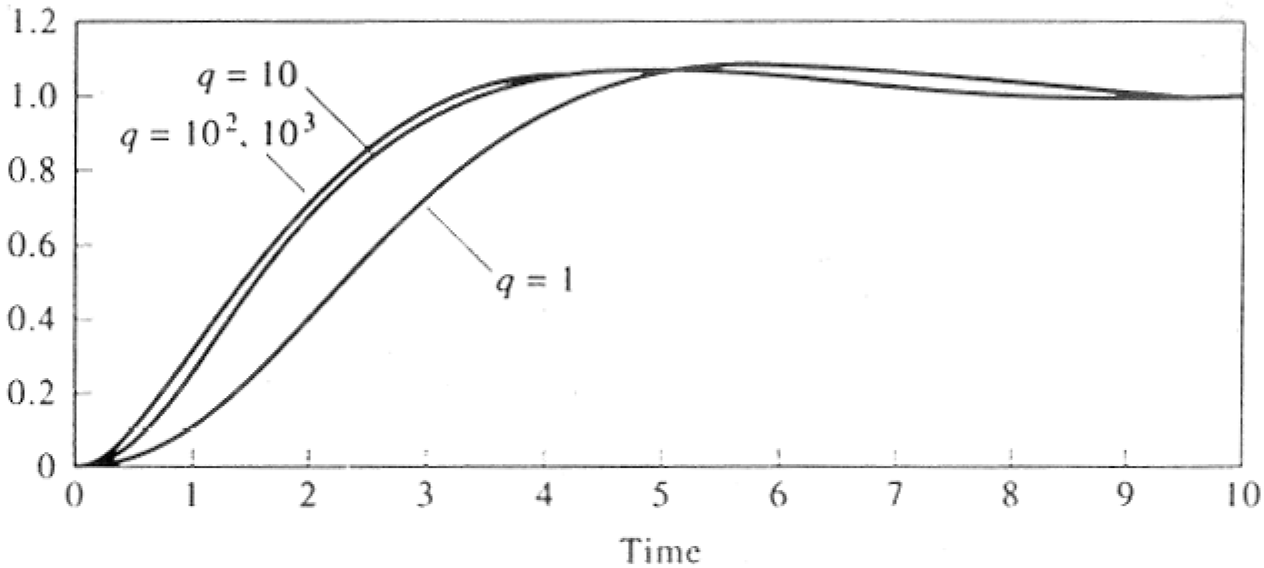
1) large gains in L are required
2) the undesirable -20dB/dec high freq. roll-off of LQR will be recovered

<u>Example</u>

Double integrator system

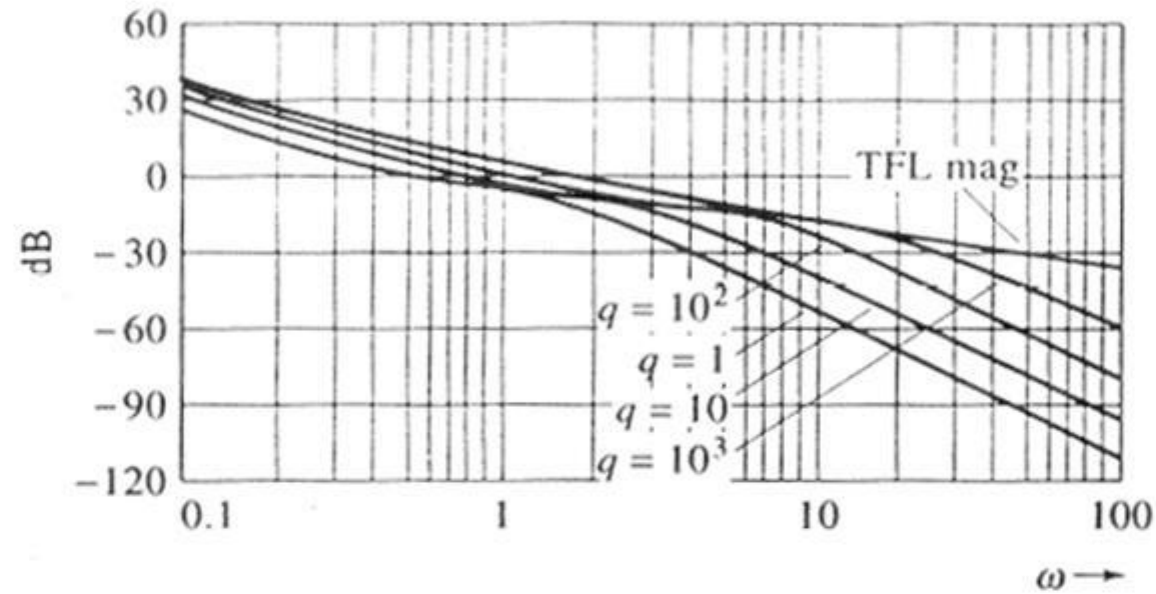
with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ R=1

→ Gave 65° phase margin for LQR design

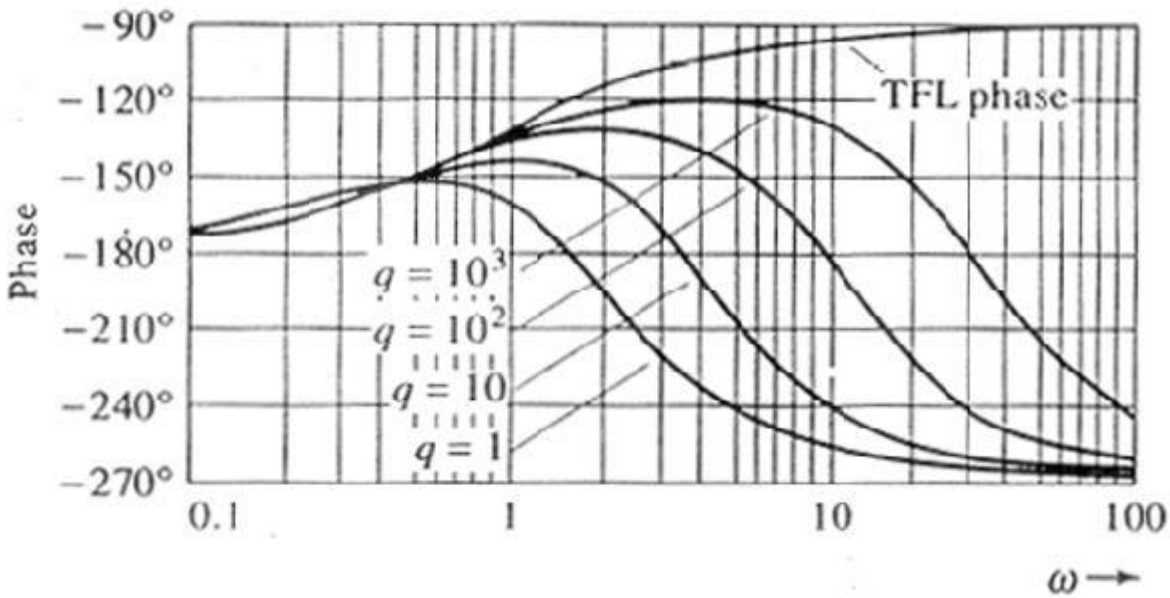


(*a*)

Figure 10.19 Step response, Bode plots, and filter poles for LTR using $q = (1, 10, 100, 1000)$. (a) Closed-loop step response. (b) and (c) Open-loop magnitude and phase Bode plots. (d) Filter poles.



(b)

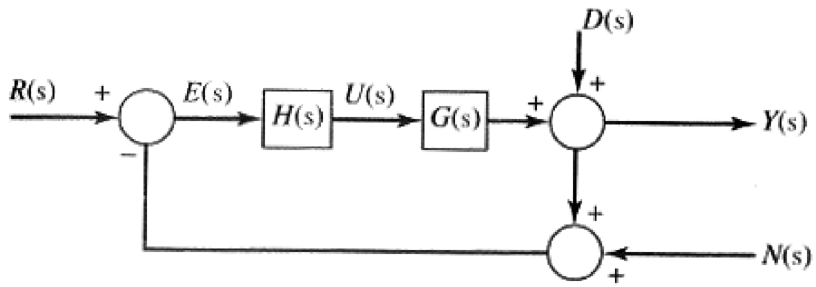


q	1	10	100	1000
PM	32.6	41.9	55.0	61.7
GM	9.5	13.0	21.1	30.4
L	1.4 1.0	4.5 10.0	14.1 100.0	44.7 1000.0
Filter poles	$-0.7 + 0.7j$ $-0.7 - 0.7j$	$-2.2 + 2.2j$ $-2.2 - 2.2j$	$-7.0 + 7.0j$ $-7.0 - 7.0j$	$-22.3 + 22.3j$ $-22.3 - 22.3j$

Robustness

1) Robust stability – stable in the face of plant uncertainties
2) Robust performance – performance met even in the face of plant uncertainties

Two important properties of feedback –
1) sensitivity reduction
2) disturbance rejection



General feedback system

$$Y(s) = \frac{G(s)H(s)}{1+G(s)H(s)}R(s) + \frac{1}{1+G(s)H(s)}D(s) - \frac{G(s)H(s)}{1+G(s)H(s)}N(s)$$

Tracking error $e = r - y$

→ $$E(s) = \frac{1}{1+G(s)H(s)}R(s) - \frac{1}{1+G(s)H(s)}D(s) - \frac{1}{1+G(s)H(s)}N(s)$$

Actuator output (i.e. plant input) is given by

$$U(s) = \frac{H(s)}{1+G(s)H(s)}\left[R(s) - D(s) - N(s)\right]$$

note: $U(s) = H(s)E(s)$
$\Rightarrow E(s) = H^{-1}(s)U(s)$

Define the following terms

$J(s) = 1 + GH$ return difference

$S(s) = \frac{1}{1+GH}$ sensitivity

$T(s) = \frac{GH}{1+GH}$ complementary sensitivity

note: $S(s) + T(s) = 1$

Using these definitions

system output: $Y(s) = S(s)D(s) + T(s)\left[R(s) - N(s)\right]$

tracking error: $E(s) = S(s)\left[R(s) - D(s) - N(s)\right]$

plant input: $U(s) = H(s)S(s)\left[R(s) - D(s) - N(s)\right]$

From these expressions we see that we need

1) Disturbance rejection: From Y(s) expression we see we require S small → GH>>1 (since SD)

2) Tracking: S small

3) Noise suppression: From Y(s) we have T(s)N(s) → require T small

4) Actuator limits: From U(s) expression want H(s)S(s) bounded

Tracking and Disturbance rejection require small S
Noise suppression requires small T

however S + T = 1
however command inputs and disturbances are low frequency whereas measurement noise is high frequency signal

→ keep S small in low frequency range and T small in high frequency range

Also $$H(s)S(s) = \frac{H(s)}{1+G(s)H(s)} = \frac{T(s)}{G(s)}$$

→ making T small we reduce control energy

Loop Gain Properties

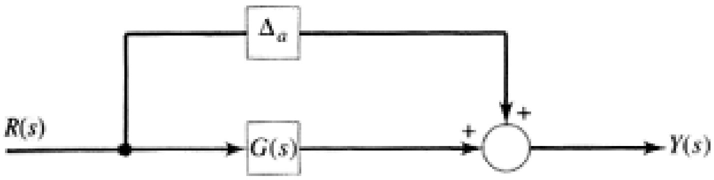
	Low Frequency	Mid. Frequency	High Frequency
Performance (R)	High Gain	Smooth Transition (for good margins)	
Disturbance Rejection (D)	High Gain		
Noise Suppression (N)			Low Gain

Uncertainty Modeling

Two categories -- 1) structured uncertainty
2) unstructured uncertainty

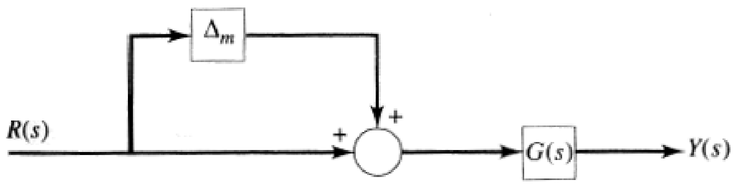
We will deal with unstructured uncertainty

Additive uncertainty: actual model $\tilde{G}(s)$

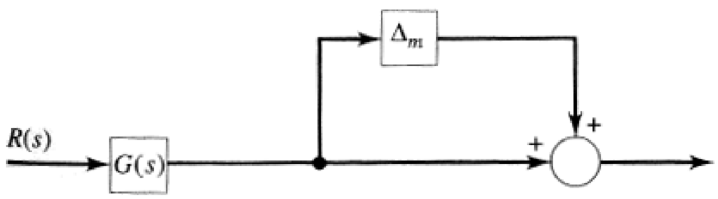


$$\tilde{G}(s) = \underbrace{G(s)}_{\text{model}} + \underbrace{\Delta_a(s)}_{\text{uncertainty or error}}$$

Multiplicative uncertainty: $\tilde{G}(s) = [1 + \Delta_m(s)]G(s)$



input uncertainty



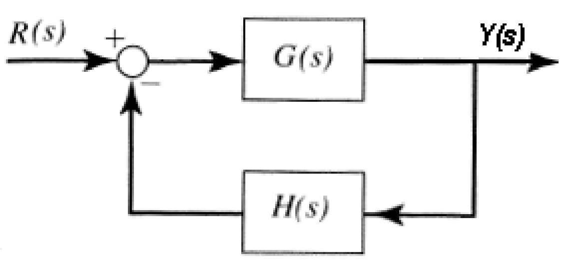
output uncertainty

Robust Stability

We say a compensator robustly stabilizes a system if the closed-loop system remains stable for the true plant $\tilde{G}(s)$.

Robustness results can be derived using the small gain theorem.

Small Gain Theorem



The closed-loop system will remain stable if

$$|G(s)H(s)| < 1$$

no since $|G(s)H(s)| \le |G(s)||H(s)|$

then closed-loop stability is guaranteed if

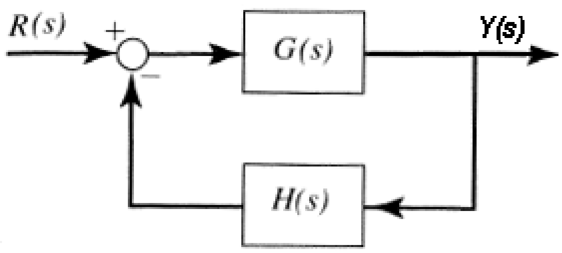
$$|G(s)H(s)| < 1$$

There is no possibility of encirclements of (-1,0) point by Nyquist plot.

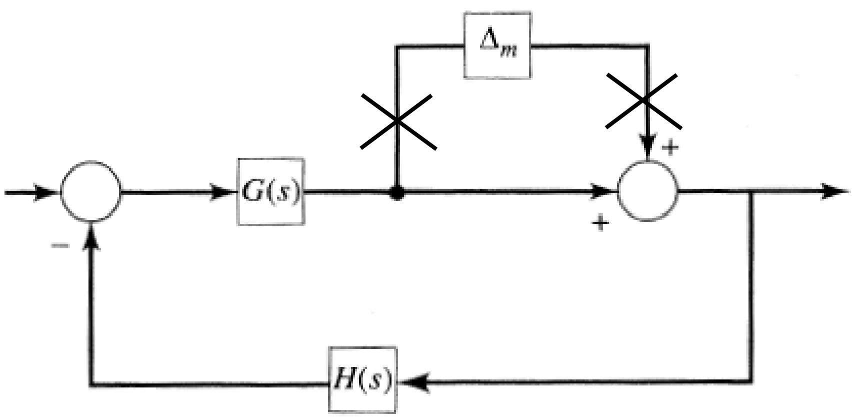
Two equations that the small gain theorem can help us to answer

1) Given that the uncertainty is stable and bounded, will the closed-loop system be stable for the given uncertainty?
2) For a given system, what is the smallest uncertainty that will destabilize the system?

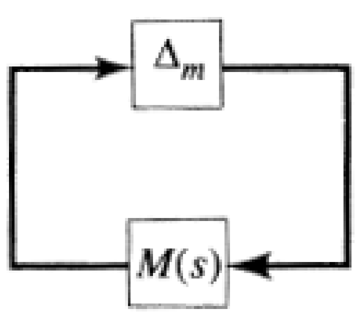
To answer these questions we first do some block diagram manipulation



With multiplicative output uncertainty →



where $M(s) = \dfrac{-G(s)H(s)}{1+G(s)H(s)}$



Determine M(s), the transfer function seen by Δ_m

By small gain theorem, closed-loop system will be robustly stable if

$$|\Delta_m| < \frac{1}{|GH(1+GH)^{-1}|}$$

i.e. $|\Delta_m| < \frac{1}{|T|}$ T – complementary sensitivity

If the uncertainty is bounded by γ so that

$$|\Delta_m| < \gamma$$

then the closed-loop system will be stable if

$$|T| < \frac{1}{\gamma} \quad \text{or} \quad |\gamma T| < 1$$

This answers the first question

Second question: find the size of the smallest stable uncertainty that will destabilize the system

Because the uncertainty must be smaller that 1/T, it must be smaller that the minimum of 1/T. We must find the maximum of T.

Define $M_r = \sup_{\omega} |T(j\omega)|$ sup = supremum (least upper bound)

Then the smallest destabilizing uncertainty, we call this the multiplicative stability margin or MSM, is given by

$$MSM = \frac{1}{M_r}$$

For additive uncertainty

$$M(s) = \frac{-H(s)}{1+G(s)H(s)}$$

closed-loop will be robustly stable if

$$|\Delta_a| < \frac{1}{|H(1+GH)^{-1}|} \quad \text{or} \quad |\Delta_a| < \frac{1}{|HS|}$$

if uncertainty is stable and bounded by

$$|\Delta_a| < \gamma$$

then we guarantee closed-loop stability if

$$|HS| < \frac{1}{\gamma} \qquad \text{or} \qquad |\gamma HS| < 1$$

we can define <u>additive stability margin</u> (ASM) by

$$ASM = \frac{1}{\sup_{\omega} |H(j\omega)S(j\omega)|}$$

<u>Example</u>

$$G(s) = \frac{5-s}{(s+5)(s^2+0.2s+1)} \quad , \qquad H(s) = \frac{5(s+0.1)}{s}\frac{s+0.2}{s+5}$$

phase margin: 38°
gain margin: 2.8 (9dB)

Find MSM and ASM:

<u>MSM</u>

Find peak of T (complementary sensitivity function)

peak = 1.52 → MSM = 0.65

→ the system will be robustly stable against unmodelled multiplicative uncertainties with transfer function magnitude < 0.65

Problem 10.9

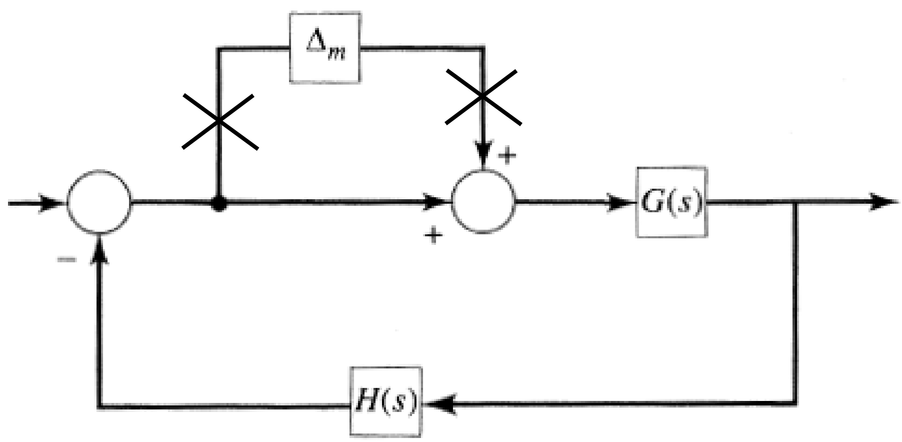
a.) $\tilde{G} = (1+\Delta_m)G \quad \rightarrow \quad \Delta_m = \dfrac{\tilde{G}}{G} - 1$

$$\Delta_m = \frac{\dfrac{2(s+1)}{s^2(s^2+s+1)}}{\dfrac{1}{s^2}} - 1$$

$$= \frac{2(s+1)}{s^2(s^2+s+1)} - \frac{s^2+s+1}{s^2(s^2+s+1)}$$

$$= \frac{-s^2+s+1}{s^2+s+1}$$

b.)



M(s) $$\frac{-GH}{(1+GH)} = \frac{-\dfrac{20(s+1)}{s^2(s+10)}}{1+\dfrac{20(s+1)}{s^2(s+10)}} = \frac{-20(s+1)}{s^2(s+10)+20(s+1)} = \frac{20(s+1)}{s^3+10s^2+20s+20}$$

c.) SGT: $|\Delta_m||M| < 1$

$$\Rightarrow |\Delta_m| < \frac{1}{|M|} = \frac{1}{\dfrac{GH}{1+GH}}$$

$$\Rightarrow |\Delta_m| < \left|1+(GH)^{-1}\right|$$

Additive uncertainty

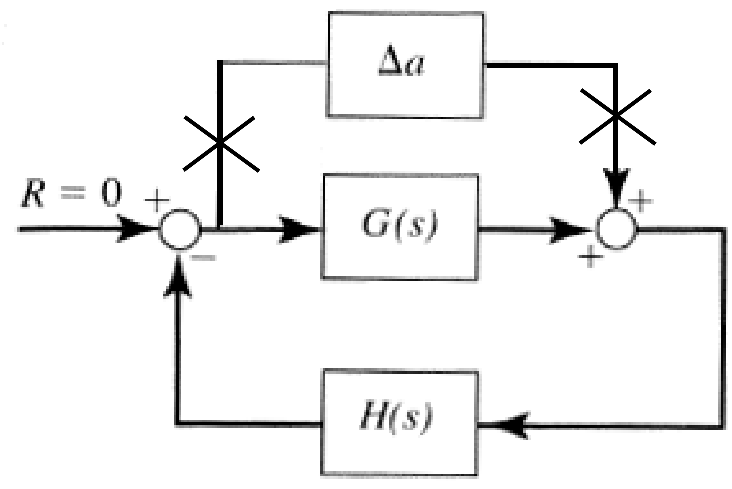
$$\tilde{G} = G + \Delta_a$$

$$\Rightarrow \Delta_a = \tilde{G} - G$$

$$= \frac{2(s+1)}{s^2(s^2+s+1)} - \frac{1}{s^2}$$

$$= \frac{1}{s^2}\left[\frac{2s+2-s^2-s-1}{s^2+s+1}\right]$$

$$= \frac{-s^2+s+1}{s^2(s^2+s+1)}$$



$$M(s) = \frac{-H}{(1+GH)}$$

SGT: $|\Delta_a||M| < 1$

$$\Rightarrow |\Delta_a| < \frac{1}{|M|}$$

$$\Rightarrow |\Delta_a| < |H^{-1} + G|$$

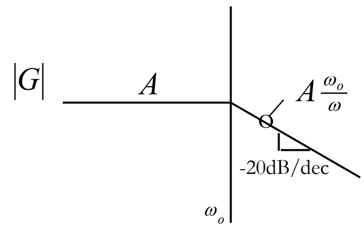
$$|\Delta_a| < \left|\frac{s+10}{20s+20} + \frac{1}{s^2}\right|$$

$$< \frac{s^2(s+10)+20s+20}{s^2(20s+20)}$$

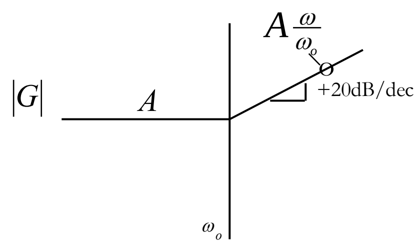
$$|\Delta_a| < \frac{s^3+10s^2+20s+20}{s^2(20s+20)}$$

Basic Bode Magnitude Plots

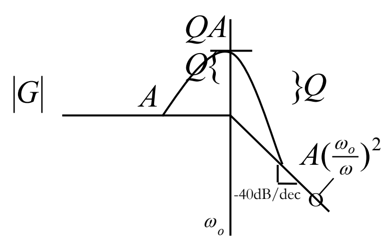
$$G(s) = \frac{A}{1+\frac{s}{\omega_0}} \Rightarrow$$



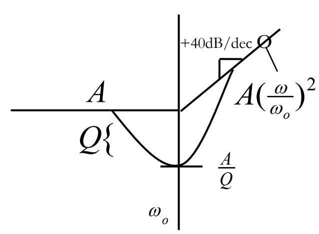
$$G(s) = A(1+\tfrac{s}{\omega_0}) \Rightarrow$$



$$G(s) = \frac{A}{1+\frac{1}{Q}(\frac{s}{\omega_0})+(\frac{s}{\omega_0})^2} \Rightarrow$$



$$G(s) = A\left[1+\tfrac{1}{Q}(\tfrac{s}{\omega_0})+(\tfrac{s}{\omega_0})^2\right] \Rightarrow$$

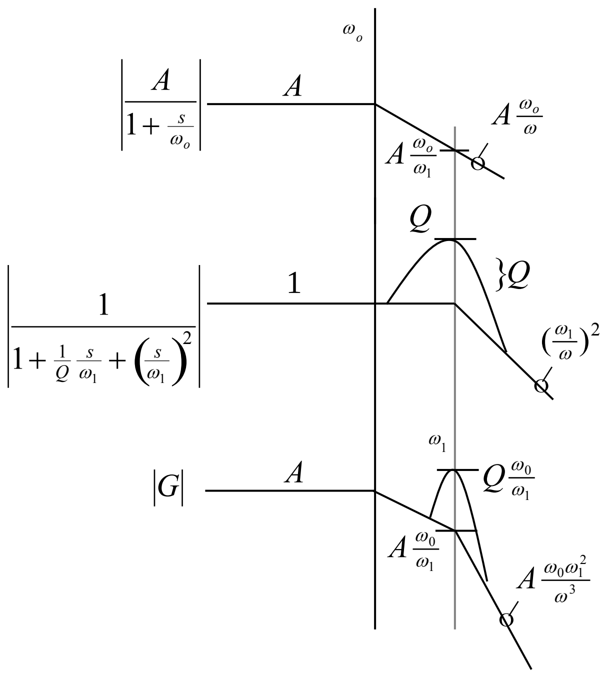


If Q < $\frac{1}{2}$ then roots are real. Factor the expression and use the resulting product of two first order transfer functions to find magnitude response.

Example

$$G(s) = \frac{A}{(1+\frac{s}{\omega_0})\left[1+\frac{1}{Q}(\frac{s}{\omega_1})+(\frac{s}{\omega_1})^2\right]} \Rightarrow$$

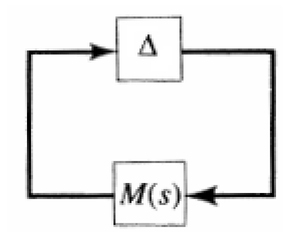
$$\omega_0 < \omega_1$$



Example

The ΔM structure of a system has been determined to be given by

$$\Delta = \frac{A_\Delta}{1+\frac{1}{Q}(\frac{s}{\omega_a})+(\frac{s}{\omega_a})^2}$$



$$M = \frac{A_M(1+\frac{s}{\omega_2})}{(1+\frac{s}{\omega_1})(1+\frac{s}{\omega_3})}$$

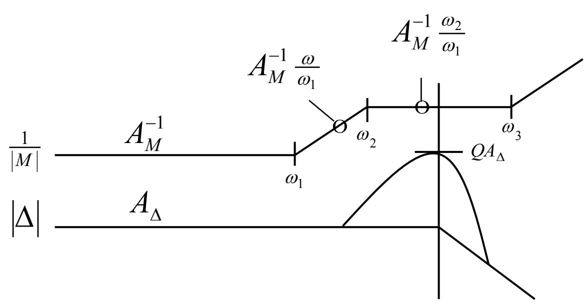
where $\omega_1 << \omega_2 << \omega_3$ and $\omega_a = \sqrt{\omega_2\omega_3}$

Determine the conditions under which robust stability is assured.

Answer

By SGT we require $|\Delta||M| < 1$ or $|\Delta| < \frac{1}{|M|}$

$$\frac{1}{M} = \frac{A_M^{-1}(1+\frac{s}{\omega_1})(1+\frac{s}{\omega_3})}{(1+\frac{s}{\omega_2})}$$



From the above diagram we can see that we require

- $A_\Delta < A_M^{-1}$ and $QA_\Delta < A_M^{-1}\frac{\omega_2}{\omega_1}$

or

- $A_\Delta < A_M^{-1}\frac{\omega_2}{Q\omega_1}$