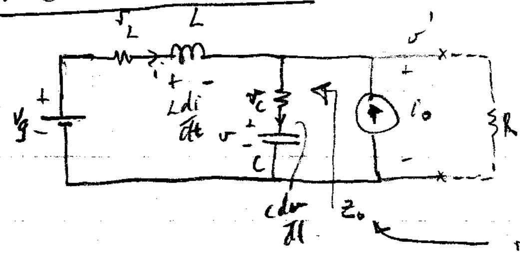
VOLTAGE MODE DROOP CONTROL ANALYSIS

DT_s



represents output impedance without R connected

THE ANALYSIS PRESENTED HERE REFERS TO THE PAPER "DESIGN CONSIDERATIONS FOR VRM TRANSIENT RESPONSE BASED ON THE OUTPUT IMPEDANCE" by K. YAO et al.

$$L\frac{di}{dt} = -(r_c + r_L)\, i - v + v_g - r_c i_o$$

$$C\frac{dv}{dt} = i + i_o$$

$$v' = r_c i + v + r_c i_o.$$

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{(r_c+r_L)}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} i \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L} & -\frac{r_c}{L} \\ 0 & \frac{1}{C} \end{bmatrix}}_{B_1} \begin{bmatrix} v_g \\ i_o \end{bmatrix}$$

$$A_1 = A_2 = A \qquad B_2 = \begin{bmatrix} 0 & -\frac{r_c}{L} \\ 0 & \frac{1}{C} \end{bmatrix}$$

$$y = v' = \underbrace{\begin{bmatrix} r_c & 1 \end{bmatrix}}_{C_1} \begin{bmatrix} i \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & r_c \end{bmatrix}}_{E_1} \begin{bmatrix} v_g \\ i_o \end{bmatrix}$$

$$C_1 = C_2 = C \qquad E_1 = E_2 = E$$

$$Z_o = \frac{\hat{v}'}{\hat{i}_o} = C\,(sI - A)^{-1} B_{i_o} + E_{i_o}$$

$$= \begin{bmatrix} r_c & 1 \end{bmatrix} \begin{bmatrix} s + \frac{r_c + r_L}{L} & \frac{1}{L} \\ -\frac{1}{C} & s \end{bmatrix}^{-1} \begin{bmatrix} -\frac{r_c}{L} \\ \frac{1}{C} \end{bmatrix} + r_c$$

$$\Delta = |sI - A| = s^2 + s\frac{(r_c + r_L)}{L} + \frac{1}{LC} = \frac{1}{LC}\left(LCs^2 + s(r_c + r_L)C + 1\right)$$

$$Z_o = \frac{1}{\Delta}\begin{bmatrix} r_c & 1 \end{bmatrix}\begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s + \frac{r_c + r_L}{L} \end{bmatrix}\begin{bmatrix} -\frac{r_c}{L} \\ \frac{1}{C} \end{bmatrix} + r_c$$

$$= \frac{1}{\Delta}\begin{bmatrix} r_c s + \frac{1}{C} & -\frac{r_c}{L} + s + \frac{r_c + r_L}{L} \end{bmatrix}\begin{bmatrix} -\frac{r_c}{L} \\ \frac{1}{C} \end{bmatrix} + r_c$$

$$= \frac{1}{\Delta}\left[\left(\frac{r_c^2}{L}s - \frac{r_c}{LC} + \frac{s}{C} + \frac{r_L}{LC}\right) + \Delta r_c\right]$$

$$= \frac{1}{\Delta}\left(\cancel{-\frac{r_c^2}{L}s} \cancel{- \frac{r_c}{LC}} + \frac{s}{C} + \frac{r_L}{LC} + r_c s^2 + \cancel{\frac{r_c^2}{L}s} + \frac{r_c r_L s}{L} + \cancel{\frac{r_c}{LC}}\right)$$

$$= \frac{1}{\Delta} r_L \left(\frac{r_c}{r_L}s^2 + \frac{s}{r_L C} + \frac{r_c s}{L} + \frac{1}{LC}\right)$$

$$= \frac{1}{\Delta}\frac{1}{LC} r_L \underbrace{\left[\frac{r_c}{r_L}LCs^2 + \left(\frac{L}{r_L} + r_c C\right)s + 1\right]}$$

$$= \left(1 + \frac{s}{\frac{r_L}{L}}\right)\left(1 + \frac{s}{\frac{1}{r_c C}}\right)$$

$$= r_L \frac{\left(1 + \frac{s}{\omega_L}\right)\left(1 + \frac{s}{\omega_c}\right)}{1 + \frac{s}{Q\omega_o} + \frac{s^2}{\omega_o^2}}$$

$$\omega_L = \frac{r_L}{L}$$

$$\omega_c = \frac{1}{r_c C}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\sqrt{L/C}}{r_L + r_c}$$

$$G_{od}(s) = C\,(sI - A)^{-1} B_d + E_d$$

$B_d = (A_1 - A_2)X + (B_1 - B_2)U = B_1 V_g$ since $A_1 = A_2$

$E_d = (C_1 - C_2)X + (E_1 - E_2)U = 0$ since $C_1 = C_2$ and $E_1 = E_2$

$$= \begin{bmatrix} r_c & 1 \end{bmatrix} \begin{bmatrix} s + \frac{r_c + r_L}{L} & \frac{1}{L} \\ -\frac{1}{C} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V_g \qquad (I_o = 0)$$

$$\Delta = |sI - A| = \frac{1}{LC}\left(LCs^2 + s(r_c + r_L)C + 1\right)$$

$$G_{od}(s) = \frac{1}{\Delta} \begin{bmatrix} r_c & 1 \end{bmatrix} \begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s + \frac{r_c + r_L}{L} \end{bmatrix} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V_g$$

$$= \frac{1}{\Delta} \begin{bmatrix} r_c s + \frac{1}{C} & \underbrace{-\frac{r_c}{L} + s + \frac{r_c + r_L}{L}}_{s + \frac{r_L}{L}} \end{bmatrix} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V_g$$

$$= \frac{1}{\Delta}\left(\frac{r_c s}{L} + \frac{1}{LC}\right) V_g$$

$$= \frac{1}{\Delta}\frac{1}{LC}\left(1 + \frac{s}{\frac{1}{r_c C}}\right) V_g$$

$$G_{od}(s) = V_g \frac{\left(1 + \frac{s}{\omega_c}\right)}{1 + \frac{s}{Q\omega_o} + \frac{s^2}{\omega_o^2}}$$

$$\omega_c = \frac{1}{r_c C}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\sqrt{L/C}}{r_L + r_c}$$

$$Z_{oc} = \frac{Z_o}{1+T} = \frac{Z_o}{1 + F_m G_{vd} G_{con}} = r_C$$

$$\Rightarrow \quad r_C = \frac{r_L \dfrac{\left(1+\frac{s}{\omega_L}\right)\left(1+\frac{s}{\omega_C}\right)}{1+\frac{s}{Q\omega_o}+\frac{s^2}{\omega_o^2}}}{1 + F_m \dfrac{\left(1+\frac{s}{\omega_C}\right) V_g}{1+\frac{s}{Q\omega_o}+\frac{s^2}{\omega_o^2}} G_{con}}$$

$$r_C = \frac{r_L \left(1+\frac{s}{\omega_L}\right)\left(1+\frac{s}{\omega_C}\right)}{\left(1+\frac{s}{Q\omega_o}+\frac{s^2}{\omega_o^2}\right) + F_m \left(1+\frac{s}{\omega_C}\right) V_g G_{con}}$$

$$\underbrace{r_C\left(1+\frac{s}{Q\omega_o}+\frac{s^2}{\omega_o^2}\right)}_{r_C + \frac{r_C s}{Q\omega_o} + \frac{r_C s^2}{\omega_o^2}} + r_C F_m V_g \left(1+\frac{s}{\omega_C}\right) G_{con} = r_L + \left(\frac{1}{\omega_L}+\frac{1}{\omega_C}\right) r_L s + \frac{r_L s^2}{\omega_L \omega_C}$$

$$\Rightarrow \quad r_C F_m V_g \left(1+\frac{s}{\omega_C}\right) G_{con} = \underbrace{r_L - r_C + \left[\left(\frac{1}{\omega_L}+\frac{1}{\omega_C}\right) r_L - \frac{r_C}{Q\omega_o}\right] s + \left(\frac{r_L}{\omega_L \omega_C} - \frac{r_C}{\omega_o^2}\right) s^2}_{} \quad (*)$$

$$= (r_L - r_C)\left(1+\frac{s}{\omega_{zr}}\right)\left(1+\frac{s}{\omega_{zr}'}\right)$$

where $$\omega_{zr} = \frac{b - \sqrt{b^2-4ac}}{2a} \qquad \omega_{zr}' = \frac{b+\sqrt{b^2-4ac}}{2a}$$

where
$$a = \frac{r_L}{\omega_L \omega_C} - \frac{r_C}{\omega_o^2}$$
$$b = r_L\left(\frac{1}{\omega_L}+\frac{1}{\omega_C}\right) - \frac{r_C}{Q\omega_o}$$
$$c = r_L - r_C$$

$$\Rightarrow G_{con} = \frac{(r_L - r_c)\left(1 + \frac{s}{\omega_{zv}}\right)\left(1 + \frac{s}{\omega'_{zv}}\right)}{r_c F_m V_g \left(1 + \frac{s}{\omega_c}\right)}$$

this higher zero is ignored

$$\Rightarrow G_{con} = K_v \frac{1 + \frac{s}{\omega_{zv}}}{1 + \frac{s}{\omega_{pv}}}$$

$$K_v = \frac{r_L - r_c}{r_c F_m V_g}$$

$$\omega_{pv} = \omega_c$$

ω_{zv} is given above.

The paper "Design Considerations for VRM Transient Response Based on the Output Impedance" leaves the analysis at the above point, namely identifying a zero at ω_{zv} (as given above) and ignoring ω'_{zv}. However, the analysis can be made more complete and accurate as follows on the next page.

MORE COMPLETE AND ACCURATE ANALYSIS

We begin by copying equation (*) from two pages ago:

$$r_C F_m V_g \left(1 + \frac{s}{\omega_C}\right) G_{con} = r_L - r_C + \left[\left(\frac{1}{\omega_L} + \frac{1}{\omega_C}\right) r_L - \frac{r_C}{Q\omega_0}\right] s + \left(\frac{r_L}{\omega_L \omega_C} - \frac{r_C}{\omega_0^2}\right) s^2 \qquad (*)$$

where

$$\omega_L = \frac{r_L}{L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega_C = \frac{1}{r_C C} \qquad Q = \frac{\sqrt{L/C}}{r_L + r_C}$$

substituting these relations into the coefficient term of s^2 of (*) we have:

$$\frac{r_L}{\omega_L \omega_C} - \frac{r_C}{\omega_0^2} = \frac{r_L}{\frac{r_L}{L}\frac{1}{r_C C}} - \frac{r_C}{\frac{1}{LC}}$$

$$= r_C (LC - LC)$$

$$= 0$$

So the quadratic in s is reduced to first order in s, and the right hand side of (*) becomes

$$= r_L - r_C + \left[\left(\frac{L}{r_L} + r_C C\right) r_L - \frac{r_C}{\frac{\sqrt{L/C}}{r_L + r_C}\frac{1}{\sqrt{LC}}}\right] s$$

$$= r_L - r_C + \Big[L + \underbrace{r_C r_L C - r_C(r_L + r_C) C}_{\cancel{r_C r_L C} - \cancel{r_C r_L C} - r_C^2 C}\Big] s$$

$$=$$

$$= r_L - r_C + (L - r_C^2 C) s$$

$$= (r_L - r_C)\left(1 + \frac{L - r_C^2 C}{r_L - r_C} s\right)$$

$$= \left(r_L - r_C\right)\left(1 + \frac{s}{\omega_Z}\right)$$

$$\text{where} \quad \omega_Z = \frac{r_L - r_C}{L - r_C^2 C}$$

Together with the left hand side of (*) we have

$$r_C F_m V_g \left(1 + \frac{s}{\omega_C}\right) G_{CON} = \left(r_L - r_C\right)\left(1 + \frac{s}{\omega_Z}\right)$$

$$\Rightarrow \quad G_{CON} = \frac{\left(r_L - r_C\right)\left(1 + \frac{s}{\omega_Z}\right)}{r_C F_m V_g \left(1 + \frac{s}{\omega_C}\right)}$$

$$= K_v \frac{\left(1 + \frac{s}{\omega_{Zv}}\right)}{1 + \frac{s}{\omega_{pv}}}$$

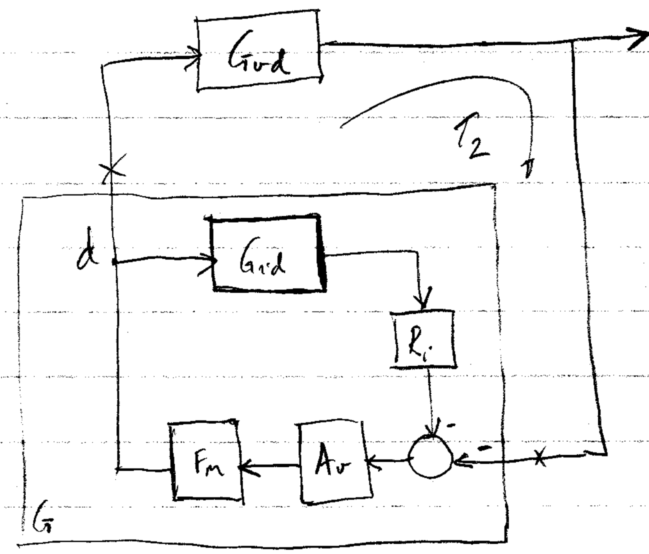
$$\text{where} \quad K_v = \frac{r_L - r_C}{r_C F_m V_g}$$

$$\omega_{pv} = \omega_C$$

$$\omega_{Zv} = \frac{r_L - r_C}{L - r_C^2 C}$$

DERIVATION OF EQNS. (17) AND (18) IN THE PAPER

"OPTIMAL DESIGN OF THE ACTIVE DROOP CONTROL METHOD FOR THE TRANSIENT RESPONSE"



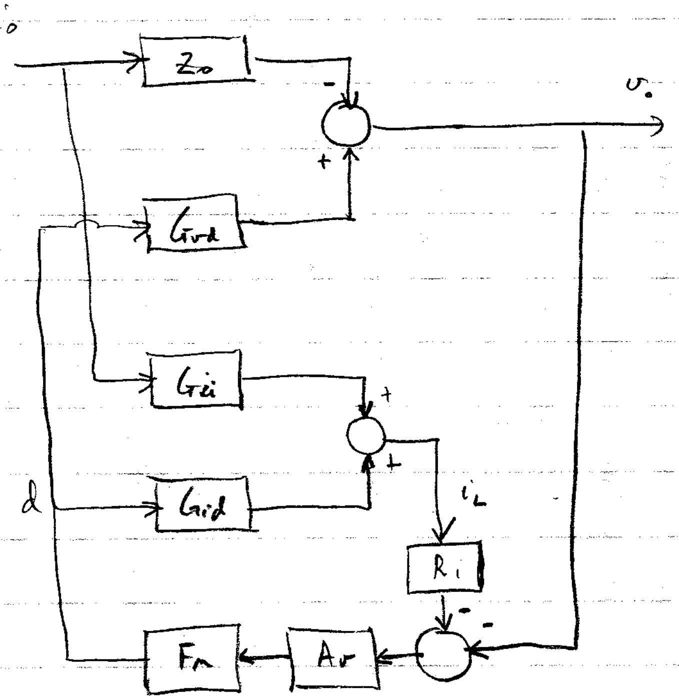
$$G = \frac{-F_m A_v}{1 + F_m A_v G_{id} R_i}$$

$$T_2 = G\, G_{vd}$$

$$= \frac{-F_m A_v G_{vd}}{1 + F_m A_v G_{id} R_i}$$

$$T_2 = \frac{-T_v}{1 + T_i} \qquad (17)$$

2



$$v_o = -Z_o i_o + G_{vd}\, d$$

$$d = -F_m A_v (v_o + R_i i_L)$$

$$i_L = G_{ii}\, i_o + G_{id}\, d.$$

$$d = -F_m A_v \left[v_o + R_i \left(G_{ii}\, i_o + G_{id}\, d \right) \right]$$

$$d = -F_m A_v v_o - F_m A_v R_i G_{ii}\, i_o - F_m A_v R_i G_{id}\, d.$$

$$d \,(1 + F_m A_v R_i G_{id}) = -F_m A_v v_o - F_m A_v R_i G_{ii}\, i_o$$

(3)

$$v_o = -Z_o i_o + G_{vd}\, d$$

$$d = \frac{-F_m A_v v_o}{1 + F_m A_v R_i G_{id}} - \frac{F_m A_v R_i G_{ii}\, i_o}{1 + F_m A_v R_i G_{id}}$$

$$= \frac{-F_m A_v v_o}{1+T_i} - \frac{F_m A_v R_i G_{ii}\, i_o}{1 + T_i}$$

$$\Rightarrow \quad v_o = -Z_o i_o - \frac{F_m A_v G_{vd}\, v_o}{1+T_i} - \frac{F_m A_v G_{vd} R_i G_{ii}\, i_o}{1+T_i}$$

$$v_o = -\left(Z_o + \frac{F_m A_v G_{vd} R_i G_{ii}}{1+T_i}\right) i_o - \frac{T_v}{1+T_i}\, v_o$$

$$\left(1 + \frac{T_v}{1+T_i}\right) v_o = -\frac{\left(Z_o \left(1+T_i\right) + F_m A_v G_{vd} R_i G_{ii}\right)}{1+T_i}\, i_o$$

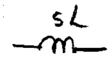
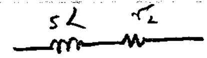
$$Z_{oc} = \frac{v_o}{i_o} = \frac{-\left(\dfrac{Z_o \left(1+T_i\right) + F_m A_v G_{vd} R_i G_{ii}}{1+T_i}\right)}{\dfrac{1+T_v+T_i}{1+T_i}}$$

$$= \frac{-\left(Z_o \left(1+T_i\right) + F_m A_v G_{vd} R_i G_{ii}\right)}{1+T_v+T_i}$$

$$Z_{oc} = \frac{-\left(Z_o \left(1+T_i\right) + \overbrace{F_m A_v G_{id} R_i}^{T_i} \cdot \dfrac{G_{vd}\, G_{ii}}{G_{id}}\right)}{1 + T_v + T_i} \qquad (19)$$

EFFECT OF PARASITICS ON THE CONTROL-TO-OUTPUT $\left(\frac{\hat{v}}{\hat{d}}(s)\right)$
TRANSFER FUNCTION OF THE BUCK CONVERTER

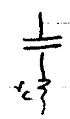
$$\frac{\hat{v}}{\hat{d}}(s) = \frac{V_g}{1 + \frac{sL}{R} + s^2 LC}$$

Effect of ESR of L :  sL $\longrightarrow$  $sL + r_L$

$$\frac{\hat{v}}{\hat{d}}(s) = \frac{V_g}{1 + \frac{sL + r_L}{R} + (sL + r_L)sC}$$

$$= \frac{V_g}{1 + \frac{r_L}{R} + \frac{sL}{R} + s r_L C + s^2 LC}$$

$$\approx \frac{V_g}{1 + s\left(\frac{L}{R} + r_L C\right) + s^2 LC} \qquad \text{for } r_L \ll R$$

Effect of ESR of C :  $\frac{1}{sC}$ $\longrightarrow$  $\frac{1}{sC} + r_C = \frac{1 + s r_C C}{sC}$

$$\Rightarrow sC \rightarrow \frac{sC}{1 + s r_C C}$$

$$\frac{\hat{v}}{\hat{d}}(s) = \frac{V_g}{1 + s\frac{L}{R} + r_L\left(\frac{sC}{1 + s r_C C}\right) + sL\frac{sC}{1 + s r_C C}}$$

$$= V_g \frac{1 + s r_C C}{1 + s r_C C + s\frac{L}{R}(1 + s r_C C) + s r_L C + s^2 LC}$$

$$= V_g \frac{1 + s r_C C}{1 + s\left[(r_C + r_L)C + \frac{L}{R}\right] + s^2 LC\left(1 + \frac{r_C}{R}\right)}$$

$$\approx V_g \frac{1 + s r_C C}{1 + s\left[(r_C + r_L)C + \frac{L}{R}\right] + s^2 LC} \qquad r_C \ll R$$