The State - Space Averaging Method

A general method for the dc and ac analysis of any Switched - mode converter

The basic state - space-averaged model

Given: a switched - mode converter consisting of linear electrical elements switched between two network topologies. Each network may be described by linear state equations:

Switch in position 1

$$\frac{d\underline{x}(t)}{dt} = A_1 \underline{x}(t) + B_1 \underline{u}(t) \qquad (1)$$

$$\underline{y}(t) = C_1 \underline{x}(t) + E_1 \underline{u}(t)$$

Switch in position 2

$$\frac{d\underline{x}(t)}{dt} = A_2 \underline{x}(t) + B_2 \underline{u}(t) \qquad (2)$$

$$\underline{y}(t) = C_2 \underline{x}(t) + E_2 \underline{u}(t)$$

Then provided that the natural converter, regulator, and line variation frequencies are much less than the switching frequency, then the large signal dc mathematical model for the converter is.

$$\underline{0} = A\underline{X}_o + B\underline{U}_o \qquad (3)$$

$$\underline{Y}_o = C\underline{X}_o + E\underline{U}_o$$

where

$$A = D_o A_1 + D_o' A_2$$

$$B = D_o B_1 + D_o' B_2$$

$$C = D_o C_1 + D_o' C_2 \qquad (4)$$

$$E = D_o E_1 + D_o' E_2$$

and

$$\underline{X_0} = \text{quiescent state vector}$$
$$\underline{U_0} = \text{" input "}$$
$$\underline{Y_0} = \text{" output "} \quad (5)$$
$$D_0 = \text{" duty "}$$

Also, the small-signal ac model is:

$$\frac{d\underline{\hat{x}}(t)}{dt} = A\underline{\hat{x}} + B\underline{\hat{u}} + [(A_1 - A_2)\underline{X_0} + (B_1 - B_2)\underline{U_0}]\,\hat{d}$$
$$\underline{\hat{y}}(t) = C\underline{\hat{x}} + E\underline{\hat{u}} + [(C_1 - C_2)\underline{X_0} + (E_1 - E_2)\underline{U_0}]\,\hat{d} \quad (6)$$

where $\underline{\hat{x}}$, $\underline{\hat{y}}$, $\underline{\hat{u}}$ and $\hat{d}$ are small ac variations about the quiescent operating point.

1) Derivation

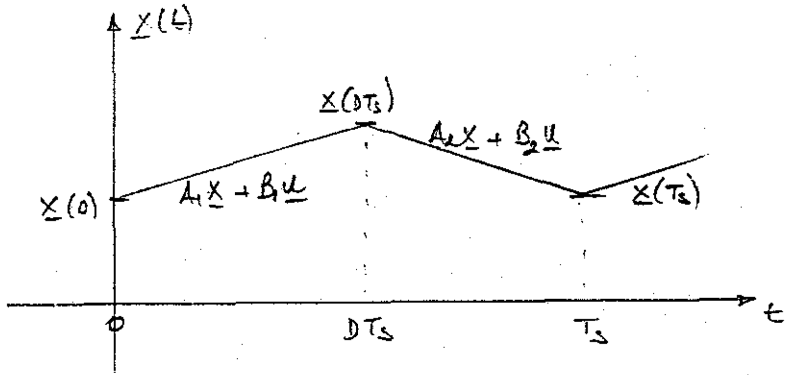
With the switch in position 1, we have

$$\frac{d\underline{x}(t)}{dt} = A_1\underline{x}(t) + B_1\underline{u}(t) \quad (7)$$

In other words, the elements of $\underline{x}(t)$ change with slope $A_1\underline{x} + B_1\underline{u}$. If we make the linear ripple approximation, i.e., that $\underline{x}(t)$ and $\underline{u}(t)$ do not change much over one switching period, then the slope is essentially constant. In essence, this coincides with the requirement for small switching ripple in all the elements of $\underline{x}(t)$, and that variations in $\underline{u}(t)$ be slow compared to the switching frequency.

Hence, if the slope is constant, we can write

$$\underline{x}(DT_s) = \underline{x}(0) + \underbrace{(DT_s)}_{\text{time}}\underbrace{(A_1\underline{x} + B_1\underline{u})}_{\text{slope}}$$



The same arguments apply during the second interval. With the switch in position 2, we have

$$\frac{d\underline{x}(t)}{dt} = A_2\underline{x} + B_2\underline{u} \qquad (9)$$

If $\underline{x}$ and $\underline{u}$ do not change much during the second interval, then

$$\underline{x}(T_s) = \underline{x}(DT_s) + \underbrace{(D'T_s)}_{\text{time}}\underbrace{(A_2\underline{x} + B_2\underline{u})}_{\text{slope}} \qquad (10)$$

Subtitution of eq. (8) into eq(10) now allows us to determine $\underline{x}(T_s)$ in terms of $\underline{x}(0)$:

$$x(T_s) = \underline{x}(0) + DT_s(A_1\underline{x} + B_1\underline{u}) + D'T_s(A_2\underline{x} + B_2\underline{u}) \qquad (11)$$

Collecting terms,

$$\underline{x}(T_s) = \underline{x}(0) + T_s(DA_1 + D'A_2)\,\underline{x} + T_s(DB_1 + D'B_2)\,\underline{u} \qquad (12)$$

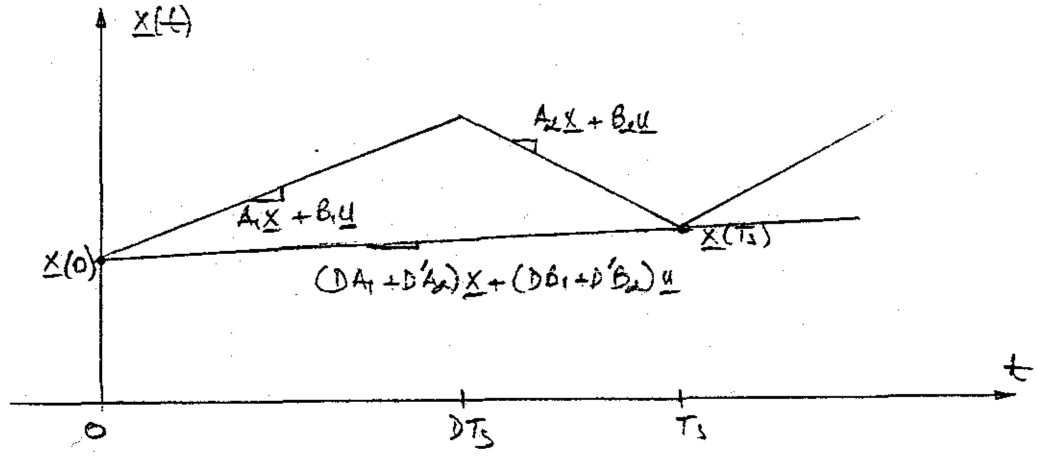
Next, we approximate the derivative:

$$\frac{d\underline{x}}{dt} = \frac{\Delta \underline{x}}{\Delta t} = \frac{\underline{x}(T_s) - \underline{x}(0)}{T_s}$$

Rearrangement of eq (12) yields

$$\boxed{\frac{d\underline{x}(t)}{dt} = \frac{\underline{x}(T_s) - \underline{x}(0)}{T_s} = (DA_1 + D'A_2)\,\underline{x} + (DB_1 + D'B_2)\,\underline{u}} \qquad 14$$

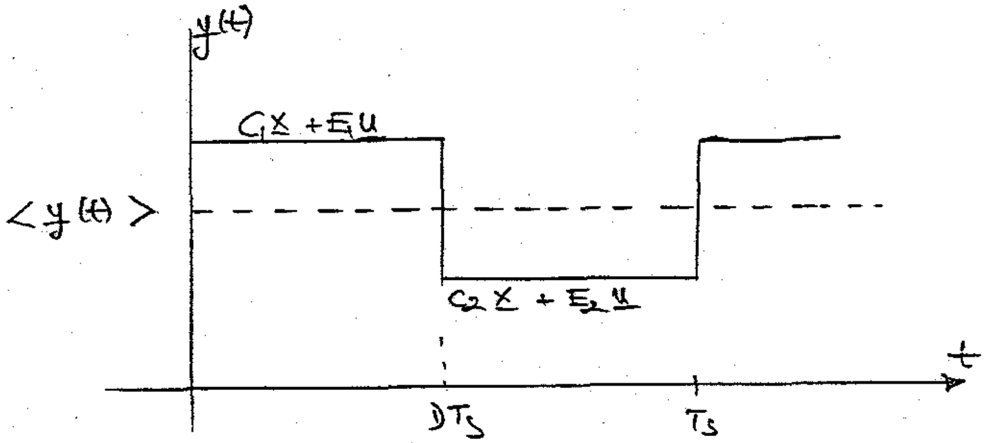
This is the basic nonlinear ac model. It is nonlinear because the time-varying quantities $D(t)$, $\underline{x}(t)$, and $\underline{u}(t)$ appear multiplied together.



<u>Output vector $\underline{y}(t)$</u>

from eqs. (1) and (2),

$$\text{output } \underline{y}(t) = \begin{cases} C_1\underline{x}(t) + E_1\underline{u}(t) & \text{during } 1^{st} \text{ interval} \\ C_2\underline{x}(t) + E_2\underline{u}(t) & \text{during } 2^{nd} \text{ interval} \end{cases}$$



If $C_1 \neq C_2$ or $E_1 \neq E_2$, then $\underline{y}(t)$ may not be a continuous function of time, we are interested in the low frequency components of $\underline{y}(t)$ (much less than switching frequency), but would like to neglect the switching harmonics.

Hence, average $y(t)$:

$$< \underline{y}(t) > = D[C_1 \underline{x} + E_1 \underline{u}] + D'[C_2 \underline{x} + E_2 \underline{u}] \quad (15)$$

Rearrangement of terms yields

$$\boxed{< \underline{y}(t) > = [DC_1 + D'C_2] \underline{x} + [DE_1 + D'E_2] \underline{u}} \quad (16)$$

Basic nonlinear averaged output relations

<u>Summary of nonlinear averaged equations:</u>

$$\frac{d\underline{x}(t)}{dt} = (DA_1 + D'A_2)\underline{x} + (DB_1 + D'B_2)\underline{u}$$

$$\underline{y}(t) = (DC_1 + D'C_2)\underline{x} + (DE_1 + D'E_2)\underline{u} \quad (17)$$

The next step is the linearization of eq. (17) about a quiescent operating point to contruct a small-signal ac model.

let

$$\underline{x}(t) = \underline{X}_o + \hat{\underline{x}}(t)$$

$$\underline{y}(t) = \underline{Y}_o + \hat{\underline{y}}(t) \quad (18)$$

$$D(t) = D_o + \hat{d}(t) \Rightarrow D'(t) = D_o' - \hat{d}(t)$$

$$\underline{u}(t) = \underline{U}_o + \hat{\underline{u}}(t)$$

where:
- X_o : Quiescent state Vector
- Y_o : ———— output Vector
- D_o : ———— duty ratio
- U_o : ———— input Vector

$\hat{d}(t)$ and $\hat{u}(t)$: are small ac variations in the duty ratio and input vector.

$\hat{x}(t)$ and $\hat{y}(t)$: are the resulting small ac variations in the state and output vectors

One must assume that these ac variations are much smaller than the quiescent values; i.e., that

$$|\hat{d}(t)| \ll D_o$$
$$\|\hat{\underline{u}}(t)\| \ll \|\underline{U}_o\| \qquad (19)$$
$$\|\hat{\underline{x}}(t)\| \ll \|X_o\|$$
$$\|\hat{\underline{y}}(t)\| \ll \|Y_o\|$$

Subtitution of eq(18) in to eq. (17) yields

$$\frac{d}{dt}\left(\underline{X}_o + \hat{\underline{x}}(t)\right) = \left[(D_o+\hat{d})A_1 + (D_o'-\hat{d})A_2\right](\underline{X}_o + \hat{\underline{x}}) + \left[(D_o+\hat{d})B_1 + (D_o'-\hat{d})B_2\right](\underline{U}_o + \hat{u}) \qquad (20)$$

$$(\underline{Y}_o + \hat{\underline{y}}) = \left[(D_o+\hat{d})C_1 + (D_o'-\hat{d})C_2\right](\underline{X}_o + \hat{\underline{x}}) + \left[(D_o+\hat{d})E_1 + (D_o'-\hat{d})E_2\right](\underline{U}_o + \hat{\underline{u}}) \qquad (21)$$

Rearrange terms:

$$\frac{d\hat{\underline{x}}}{dt} = \underbrace{(A\underline{X}_o + B\underline{U}_o)}_{DC} + \underbrace{A\hat{\underline{x}} + B\hat{\underline{u}} + \left[(A_1-A_2)\underline{X}_o + (B_1-B_2)\underline{U}_o\right]\hat{d}}_{\text{linear AC}} + \underbrace{(A_1-A_2)\hat{\underline{x}}\hat{d} + (B_1-B_2)\hat{\underline{u}}\hat{d}}_{\text{nonlinear ac}} \qquad (22)$$

$$\underline{Y}_o + \underline{\hat{y}} = \underbrace{(C\underline{X}_o + E\underline{U}_o)}_{dc} + \underbrace{C\underline{\hat{x}} + E\underline{\hat{u}} + [(C_1 - C_2)\underline{X}_o + (E_1 - E_2)\underline{U}_o]}_{\text{linear ac}}\hat{d}$$

$$+ \underbrace{(C_1 - C_2)\underline{\hat{x}}\hat{d} + (E_1 - E_2)\underline{\hat{u}}\hat{d}}_{\text{nonlinear ac}} \qquad (23)$$

where

$$A \triangleq D_o A_1 + D_o' A_2$$

$$B \triangleq D_o B_1 + D_o' B_2$$

$$C \triangleq D_o C_1 + D_o' C_2$$

$$E \triangleq D_o E_1 + D_o' E_2$$

<u>DC Model</u>

Equate dc terms

$$\underline{0} = A\underline{X}_o + B\underline{U}_o$$

$$\underline{Y}_o = C\underline{X}_o + E\underline{U}_o \qquad (24)$$

<u>Small-Signal ac Model</u>

Equate 1^{st} order ac terms

$$\frac{d\underline{\hat{x}}}{dt} = A\underline{\hat{x}} + B\underline{\hat{u}} + [(A_1 - A_2)\underline{X}_o + (B_1 - B_2)\underline{U}_o]\hat{d}$$

$$\underline{\hat{y}} = C\underline{\hat{x}} + E\underline{\hat{u}} + [(C_1 - C_2)\underline{X}_o + (E_1 - E_2)\underline{U}_o]\hat{d}$$

This is the desired result.

Appendix to lecture :

A Mathematically Rigorous Derivation of the Basic state - Space Averaging Step.

In this appendix, the averaging step of the previous derivation is justified rigorously. The basic principles used in the derivation are the same here are in the more intuitive explanation of the previous lecture but the mathematical approach taken here exposes all of the approximations used more clearly.

With the switch in position 1, we have

$$\frac{d\underline{x}(t)}{dt} = A_1 \underline{x}(t) + B_1 \underline{u}(t)$$

As decribed in Appendix L, the solution is given by

$$\underline{x}(t) = e^{A_1 t}\underline{x}(0) + e^{A_1 t}\int_0^t e^{-A_1 \tau} B_1 u(\tau)\, d\tau$$

At the end of the first interval, $t = DT_s$, and we have

$$\underline{x}(DT_s) = e^{A_1 DT_s}\underline{x}(0) + e^{A_1 DT_s}\int_0^{DT_s} e^{-A_1 \tau} B_1 u(\tau)\, d\tau$$

We next assume that variations in $\underline{u}(t)$ are much slower than the switching frequency, and hence $u(t)$ does not vary much over the interval $[0, DT_s]$. This assumption is well-satisfied in practice (for ex. V_g may vary at a 120Hz rate, while the switching frequency is 80KHz). $\underline{u}(t)$ can then be remove from the integral :

$$\underline{x}(DT_s) = e^{A_1 DT_s}\underline{x}(0) + e^{A_1 DT_s}\int_0^{D_1 T_s} e^{-A_1\tau}\, d\tau \; B_1 \underline{u}(t)$$

$$= e^{A_1 DT_s}\underline{x}(0) + e^{A_1 DT_s}\left[-A_1^{-1} e^{-A_1\tau}\right]_{\tau=0}^{D_1 T_s} B_1 \underline{u}$$

$$x(DT_s) = e^{A_1 DT_s}\underline{x}(0) + A_1^{-1}\left[e^{A_1 DT_s} - I\right] B_1 \underline{u} \qquad (*)$$

Linear Ripple Approximation:

We next assume that the converter natural frequencies are much smaller than the switching frequency, so that the exponential matrix $e^{A_1 DT_s}$ in (*) is well-approximated by the first two terms in the series expansion:

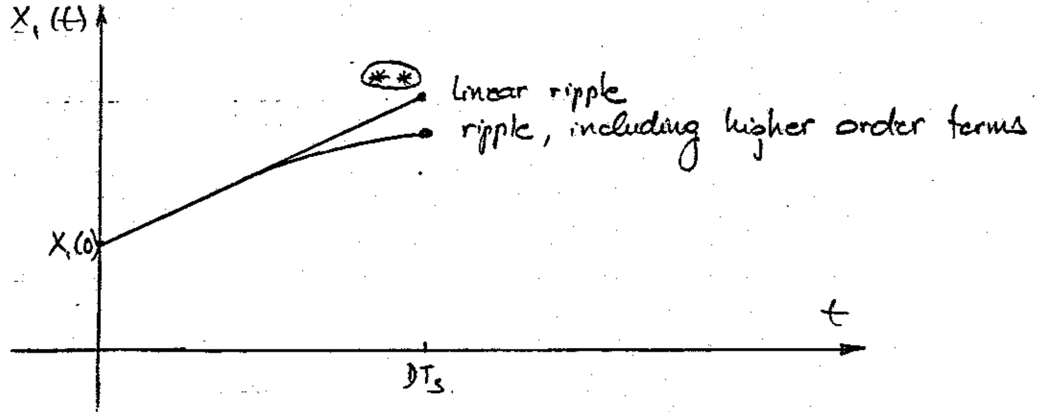
$$e^{A_1 DT_s} = I + A_1 DT_s + \frac{(A_1 DT_s)^2}{2!} + \ldots \cong I + A_1 DT_s$$

eq. (*) then becomes

$$\underline{x}(DT_s) \cong (I + A_1 DT_s)\,\underline{x}(0) + A_1^{-1}\left[I + A_1 DT_s - I\right] B_1 \underline{u}$$

or $\underline{x}(DT_s) \cong (I + A_1 DT_s)\,\underline{x}(0) + DT_s B_1 \underline{u}$ (**)

Since we have neglected terms of order $(A_1 DT_s)^2$ here, any additional second-order terms which appear in the subsequence analysis must be neglected also.



A similar analysis can be performed for the 2^{nd} interval:

$$\frac{d\underline{x}(t)}{dt} = A_2\underline{x}(t) + B_2\underline{u}(t)$$

$$\Rightarrow \underline{x}(T_s) = e^{A_2 D' T_s}\underline{x}(DT_s) + e^{A_2 D' T_s}\int_{DT_s}^{T_s} e^{-A_2\tau} B_2 u(\tau)\, d\tau$$

$$\Rightarrow \underline{x}(T_s) \cong (I + D'T_s A_2)\, x(DT_s) + D'T_s B_2 \underline{u}(t) \qquad (***)$$

Substitution of (**) into (***) yields

$$\underline{x}(T_s) \cong (I + A_2 D' T_s)\left[(I + A_1 DT_s)\underline{x}(0) + DT_s B_1 \underline{u}\right] + D'T_s B_2 \underline{u}$$

$$\cong \left[I + T_s(DA_1 + D'A_2) + T_s^2 \cancel{(DD'A_2A_1)}\right]\underline{x}(0)$$

Second order in $A_i T_s$

$$+ \left[T_s(DB_1 + D'B_2) + T_s^2 \cancel{(DD'A_2B_1)}\right]\underline{u}$$

or

$$\boxed{\underline{x}(T_s) \cong \left[I + T_s(DA_1 + D'A_2)\right]\underline{x}(0) + T_s\left[DB_1 + D'B_2\right]\underline{u}}$$

This is the basic difference equation which describes the behavior of CCM switch-mode converters. It is a discrete time system, with sampling period T_s.

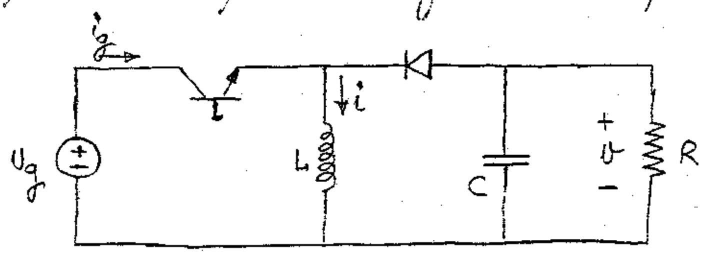
A futher approximation can be made, replacing the sampled-data system with an equivalent continuous-time system. One simply approximate the derivative by

$$\frac{d\underline{x}(t)}{dt} = \frac{\Delta \underline{x}(t)}{\Delta t} = \frac{\underline{x}(T_s) - \underline{x}(0)}{T_s} \qquad (\text{Valid to order } T_s)$$

$$= (DA_1 + D'A_2)\,\underline{x}(0) + (DB_1 + D'B_2)\,\underline{u}$$

This is the result obtained previously.

Example - state - space - Averaged Model for the Buck - Boost Converter



Include effects of transistor and diode voltage drops V_T and V_D

It is desired to determine both the output voltage v and the line current i_g

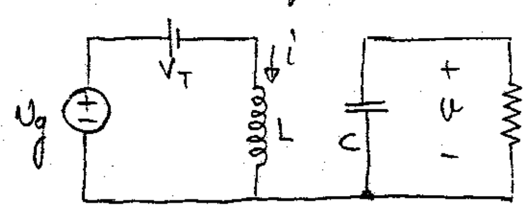
$\Rightarrow$ output vector $\underline{y}(t) = \begin{bmatrix} v(t) \\ i_g(t) \end{bmatrix}$

state vector $\underline{x}(t) = \begin{bmatrix} i(t) \\ v(t) \end{bmatrix}$

Input vector $\underline{u}(t) = \begin{bmatrix} v_g(t) \\ V_T \\ V_D \end{bmatrix}$

Now write state equations for each interval:

Switch in 1^{st} position: Transistor on, diode off



$$L\frac{di}{dt} = v_g - V_T \Rightarrow \frac{di}{dt} = \frac{v_g}{L} - \frac{V_T}{L}$$

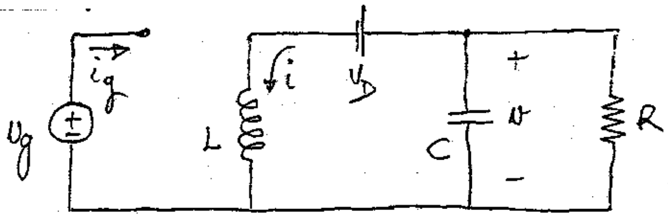
$$C\frac{dv}{dt} = -\frac{v}{R} \Rightarrow \frac{dv}{dt} = -\frac{v}{RC}$$

$$i_g = i$$

So
$$\frac{d\underline{x}}{dt} = \frac{d}{dt}\begin{bmatrix} i \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} i \\ v \end{bmatrix}}_{\underline{x}} + \underbrace{\begin{bmatrix} \frac{1}{L} & -\frac{1}{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{B_1} \underbrace{\begin{bmatrix} V_g \\ V_T \\ V_D \end{bmatrix}}_{\underline{u}}$$

$$\underline{y} = \begin{bmatrix} v \\ i_g \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{C_1} \underbrace{\begin{bmatrix} i \\ v \end{bmatrix}}_{\underline{x}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} V_g \\ V_T \\ V_D \end{bmatrix}}_{\underline{u}}$$

Switch in 2^{nd} position: transistor off, diode on



$$L\frac{di}{dt} = v - V_D \Rightarrow \frac{di}{dt} = \frac{v}{L} - \frac{V_D}{L}$$

$$C\frac{dv}{dt} = -\frac{v}{R} - i \Rightarrow \frac{dv}{dt} = -\frac{i}{C} - \frac{v}{RC}$$

$$i_g = 0$$

So we have

$$\frac{d}{dt}\begin{bmatrix} i \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}}_{A_2} \begin{bmatrix} i \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & -\frac{1}{L} \\ 0 & 0 & 0 \end{bmatrix}}_{B_2} \begin{bmatrix} V_g \\ V_T \\ V_D \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} v \\ i_g \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{C_2} \begin{bmatrix} i \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_2} \begin{bmatrix} V_g \\ V_T \\ V_D \end{bmatrix}$$

Now, the state-space averaged relations are:

DC model
$$\begin{cases} 0 = A\underline{X}_o + B\underline{U}_o \\ \underline{Y}_o = C\underline{X}_o + E\underline{U}_o \end{cases}$$

$$\frac{d\hat{\underline{x}}}{dt} = A\hat{\underline{x}} + B\hat{\underline{u}} + \left[(A_1 - A_2)\underline{X}_o + (B_1 - B_2)\underline{U}_o\right]\hat{d}$$

$$\hat{\underline{y}} = C\hat{\underline{x}} + E\hat{\underline{u}} + \left[(C_1 - C_2)\underline{X}_o + (E_1 - E_2)\underline{U}_o\right]\hat{d}$$

where
$$\underline{x}(t) = \underline{X}_o + \hat{\underline{x}}(t)$$
$$\underline{y}(t) = \underline{Y}_o + \hat{\underline{y}}(t)$$
$$\underline{u}(t) = \underline{U}_o + \hat{\underline{u}}(t)$$
$$D(t) = D_o + \hat{d}(t)$$

and
$$A \triangleq D_o A_1 + D_o' A_2$$
$$B \triangleq D_o B_1 + D_o' B_2$$
$$C \triangleq D_o C_1 + D_o' C_2$$
$$E \triangleq D_o E_1 + D_o' E_2$$

Evaluate matrices of interest:

$$A = D_o A_1 + D_o' A_2 = \begin{bmatrix} 0 & \frac{D_o'}{L} \\ -\frac{D_o'}{C} & -\frac{1}{RC} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{D_o}{L} & -\frac{D_o}{L} & -\frac{D_o'}{L} \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ D_o & 0 \end{bmatrix} \qquad E = \phi$$

$$A_1 - A_2 = \begin{bmatrix} 0 & -\frac{1}{L} \\ 1/C & 0 \end{bmatrix} \; ; \; B_1 - B_2 = \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} & -\frac{1}{L} \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_1 - C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_1 - E_2 = 0$$

$$(A_1 - A_2)\underline{X_0} + (B_1 - B_2)\underline{U_0} = \begin{bmatrix} -\frac{V_0}{L} \\ \frac{I_0}{L} \end{bmatrix} + \begin{bmatrix} \frac{V_g - V_T + V_D}{L} \\ 0 \end{bmatrix} = \begin{bmatrix} V_g - V_0 - V_T - V_D \\ \frac{I_0}{C} \end{bmatrix}$$

$$(C_1 - C_2)\underline{X_0} + (E_1 - E_2)\underline{U_0} = \begin{bmatrix} 0 \\ I_0 \end{bmatrix}$$

DC Model

$$0 = A\underline{X_0} + B\underline{U_0}$$

$$\underline{Y_0} = C\underline{X_0} + E\underline{U_0}$$

or

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{D_0'}{L} \\ -\frac{D_0'}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} I_0 \\ V_0 \end{bmatrix} + \begin{bmatrix} \frac{D_0}{L} & -\frac{D_0}{L} & -\frac{D_0'}{L} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_g \\ V_T \\ V_D \end{bmatrix}$$

$$\underline{Y_0} = \begin{bmatrix} V_0 \\ I_g \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ D_0 & 0 \end{bmatrix} \begin{bmatrix} I_0 \\ V_0 \end{bmatrix} + \underline{0}$$

We have

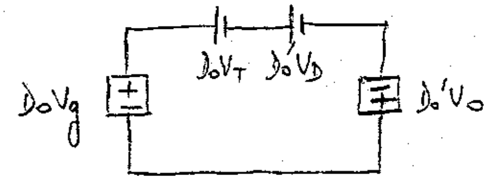
(1) $$0 = \frac{D_0'}{L} V_0 + \frac{D_0}{L} V_g - \frac{D_0}{L} V_T - \frac{D_0'}{L} V_D$$

$$\Rightarrow 0 = D_0' V_0 + D_0 V_g - D_0 V_T - D_0' V_D$$

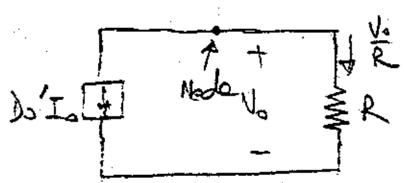
(2) $$0 = -\frac{D_0'}{C} I_0 - \frac{V_0}{RC} \Rightarrow 0 = D_0' I_0 + \frac{V_0}{R}$$

(3) $$I_g = D_0 I_0$$

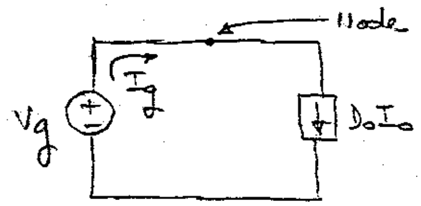
Eq. (1) : loop equation



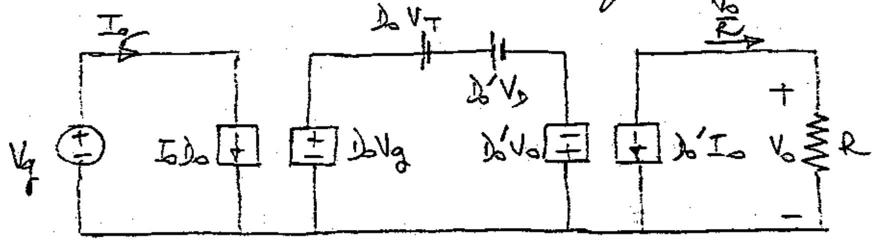
Eq. (2) : Node equation



Eq. (3) : Node equation



Combine all three circuits together :



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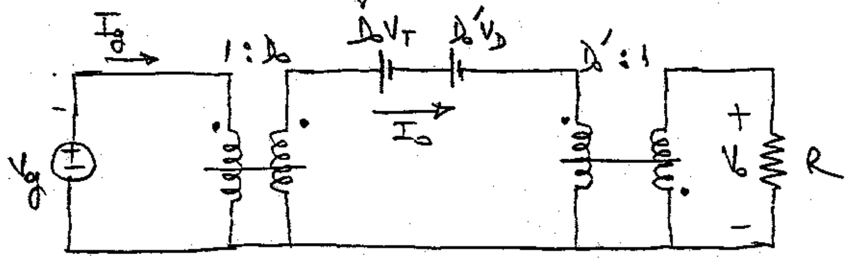


DC Model for the buck-boost Converter

AC Model

$$\frac{d\hat{\underline{x}}}{dt} = A\hat{\underline{x}} + B\hat{\underline{u}} + \left[(A_1 - A_2)\underline{X_o} + (B_1 - B_2)\underline{U_o}\right]\hat{d}$$

or

$$\frac{d}{dt}\begin{bmatrix}\hat{i}\\ \hat{v}\end{bmatrix} = \begin{bmatrix}0 & \frac{D_o'}{L}\\ -\frac{D_o'}{C} & \frac{1}{RC}\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{v}\end{bmatrix} + \begin{bmatrix}\frac{D_o}{L} & -\frac{D_o}{L} & -\frac{D_o'}{L}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\hat{v}_g\\ \cancel{\hat{v}_T} \to \emptyset\\ \cancel{\hat{v}_D} \to \emptyset\end{bmatrix}$$

$$+ \begin{bmatrix}\frac{V_g - V_o - V_T - V_D}{L}\\ I_o/C\end{bmatrix}\hat{d}$$

$$\Rightarrow \quad L\frac{d\hat{i}}{dt} = D_0'\hat{v} + D_0\hat{v}_g + (V_g - V_0 - V_T + V_D)\hat{d} \qquad (4)$$

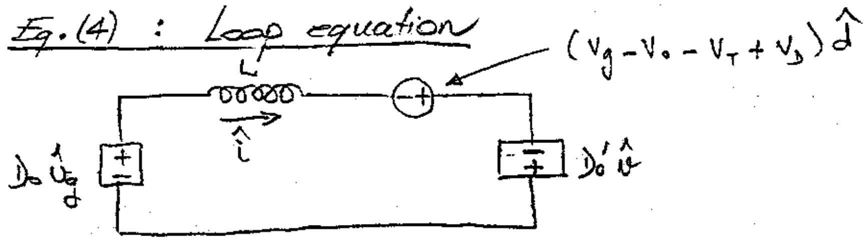
$$C\frac{d\hat{v}}{dt} = -D_0'\hat{i} - \frac{\hat{v}}{R} + I_0\hat{d} \qquad (5)$$

Output relations :

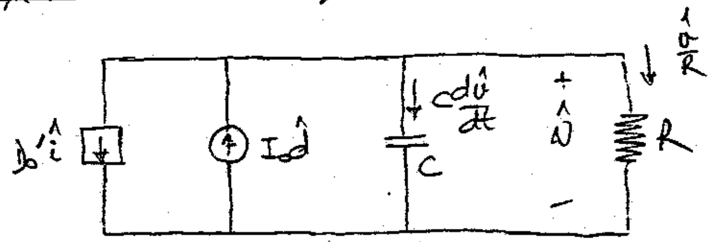
$$\begin{bmatrix} \hat{v} \\ \hat{i}_g \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ D_0 & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} 0 \\ I_0 \end{bmatrix} \hat{d}$$

$$\Rightarrow \quad \hat{i}_g = D_0\hat{i} + I_0\hat{d} \qquad (6)$$

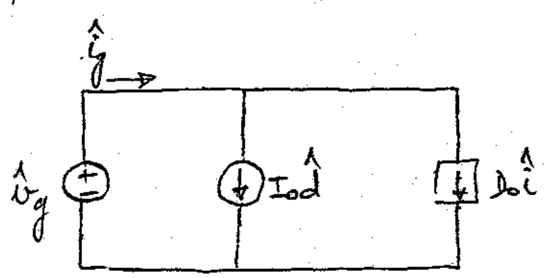
Eq. (4) : Loop equation



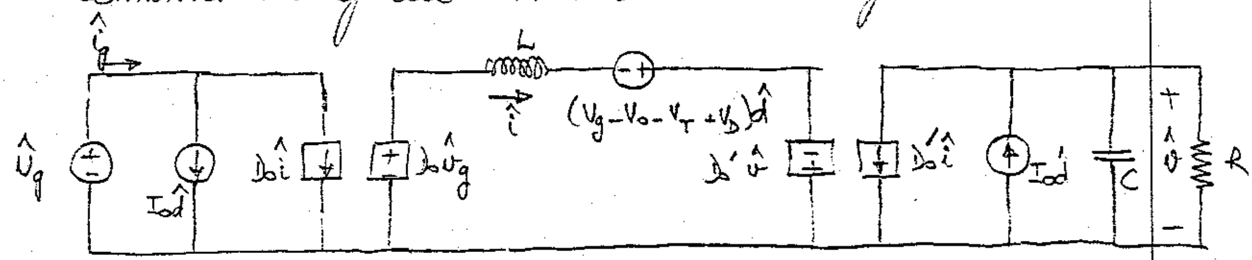
Eq. (5) : Node equation

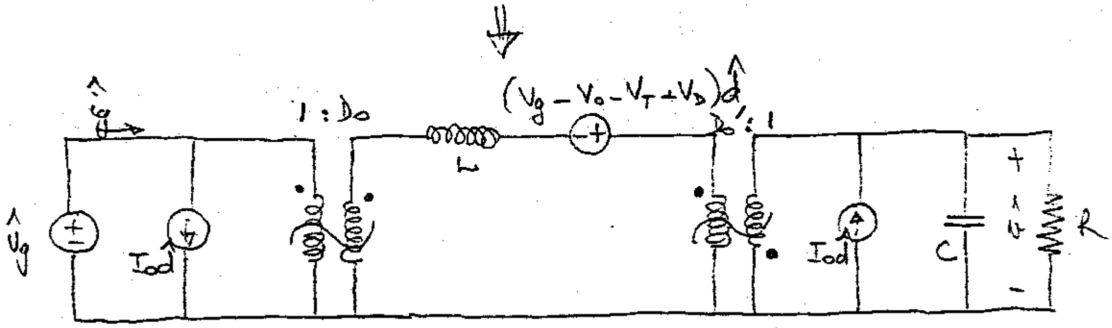


Eq. (6) : Node equation



Combination of all three ac circuits yields



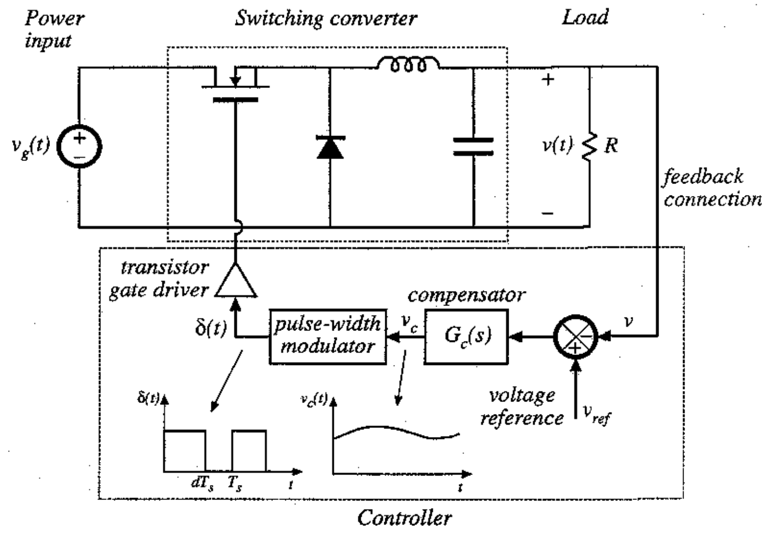


Complete ac small-signal model for the buck-boost converter

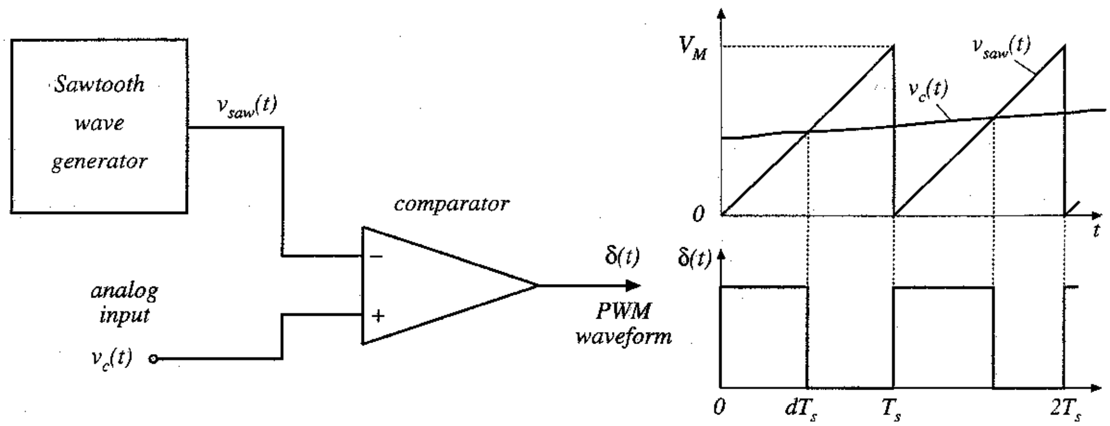
7.7. Modeling the pulse-width modulator

Pulse-width modulator converts voltage signal $v_c(t)$ into duty cycle signal $d(t)$.

What is the relation between $v_c(t)$ and $d(t)$?



A simple pulse-width modulator

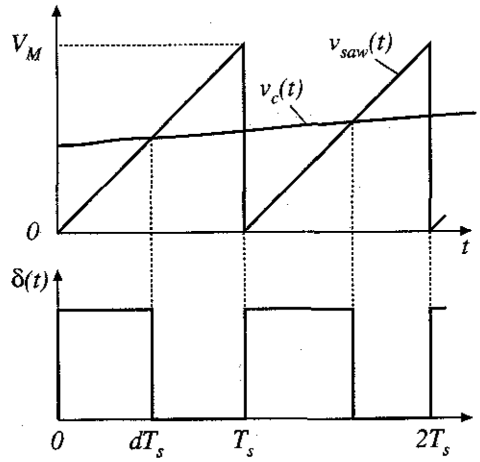


Equation of pulse-width modulator

For a linear sawtooth waveform:

$$d(t) = \frac{v_c(t)}{V_M} \qquad \text{for } 0 \le v_c(t) \le V_M$$

So $d(t)$ is a linear function of $v_c(t)$.



Perturbed equation of pulse-width modulator

PWM equation:

$$d(t) = \frac{v_c(t)}{V_M} \qquad \text{for } 0 \le v_c(t) \le V_M$$

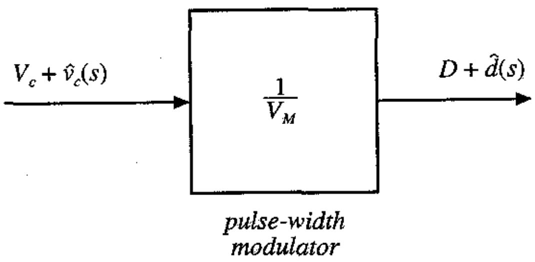
Perturb:

$$v_c(t) = V_c + \hat{v}_c(t)$$
$$d(t) = D + \hat{d}(t)$$

Result:

$$D + \hat{d}(t) = \frac{V_c + \hat{v}_c(t)}{V_M}$$

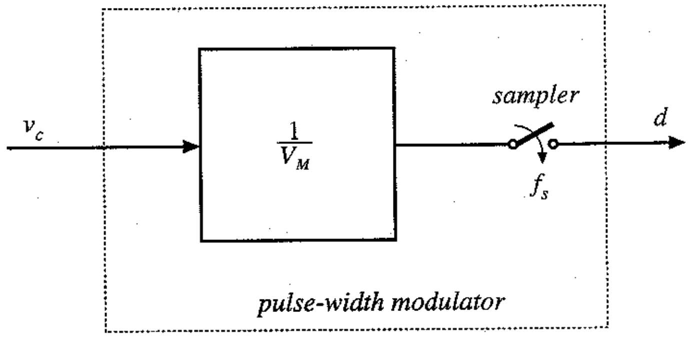
Block diagram:



Dc and ac relations:

$$D = \frac{V_c}{V_M}$$
$$\hat{d}(t) = \frac{\hat{v}_c(t)}{V_M}$$

Sampling in the pulse-width modulator



The input voltage is a continuous function of time, but there can be only one discrete value of the duty cycle for each switching period.

Therefore, the pulse-width modulator samples the control waveform, with sampling rate equal to the switching frequency.

In practice, this limits the useful frequencies of ac variations to values much less than the switching frequency. Control system bandwidth must be sufficiently less than the Nyquist rate $f_s/2$. Models that do not account for sampling are accurate only at frequencies much less than $f_s/2$.