Problem 4.1 A cube 2 m on a side is located in the first octant in a Cartesian coordinate system, with one of its corners at the origin. Find the total charge contained in the cube if the charge density is given by $\rho_{\mathrm{v}} = xy^2 e^{-2z}$ (mC/m^3).

Solution: For the cube shown in Fig. P4.1, application of Eq. (4.5) gives

$$Q = \int_{\mathscr{V}} \rho_{\mathrm{v}}\, d\mathscr{V} = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} xy^2 e^{-2z}\, dx\, dy\, dz$$
$$= \left(\frac{-1}{12} x^2 y^3 e^{-2z}\right)\Bigg|_{x=0}^{2}\Bigg|_{y=0}^{2}\Bigg|_{z=0}^{2} = \frac{8}{3}(1 - e^{-4}) = 2.62 \text{ mC}.$$

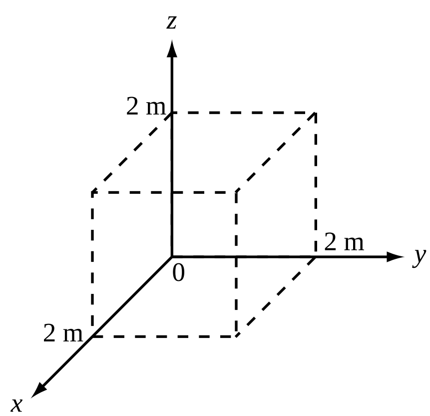

Figure P4.1: Cube of Problem 4.1.

Problem 4.5 Find the total charge on a circular disk defined by $r \le a$ and $z = 0$ if:

(a) $\rho_s = \rho_{s0} \cos\phi$ (C/m^2)

(b) $\rho_s = \rho_{s0} \sin^2\phi$ (C/m^2)

(c) $\rho_s = \rho_{s0} e^{-r}$ (C/m^2)

(d) $\rho_s = \rho_{s0} e^{-r} \sin^2\phi$ (C/m^2)

where ρ_{s0} is a constant.

Solution:

(a)

$$Q = \int \rho_s \, ds = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \cos\phi \;\; r\, dr\, d\phi = \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \left. \sin\phi \right|_0^{2\pi} = 0.$$

(b)

$$Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \sin^2\phi \;\; r\, dr\, d\phi = \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \int_0^{2\pi} \left(\frac{1 - \cos 2\phi}{2} \right) d\phi$$
$$= \frac{\rho_{s0} a^2}{4} \left. \left(\phi - \frac{\sin 2\phi}{2} \right) \right|_0^{2\pi} = \frac{\pi a^2}{2} \rho_{s0}.$$

(c)

$$Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} r\, dr\, d\phi = 2\pi \rho_{s0} \int_0^a r e^{-r}\, dr$$
$$= 2\pi \rho_{s0} \left[-r e^{-r} - e^{-r} \right]_0^a$$
$$= 2\pi \rho_{s0} [1 - e^{-a}(1 + a)].$$

(d)

$$Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} \sin^2\phi \;\; r\, dr\, d\phi$$
$$= \rho_{s0} \int_{r=0}^{a} r e^{-r}\, dr \int_{\phi=0}^{2\pi} \sin^2\phi \, d\phi$$
$$= \rho_{s0} [1 - e^{-a}(1 + a)] \cdot \pi = \pi \rho_{s0} [1 - e^{-a}(1 + a)].$$

Problem 4.6 If $\mathbf{J} = \hat{\mathbf{y}}4xz$ (A/m^2), find the current I flowing through a square with corners at $(0,0,0)$, $(2,0,0)$, $(2,0,2)$, and $(0,0,2)$.

Solution: Using Eq. (4.12), the net current flowing through the square shown in Fig. P4.6 is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_{x=0}^{2} \int_{z=0}^{2} (\hat{\mathbf{y}}4xz)\bigg|_{y=0} \cdot (\hat{\mathbf{y}}\, dx\, dz) = \left(x^2 z^2\right)\bigg|_{x=0}^{2}\bigg|_{z=0}^{2} = 16 \text{ A}.$$

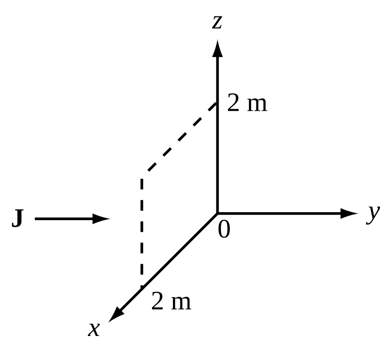

Figure P4.6: Square surface.

Problem 4.10 A line of charge of uniform density ρ_ℓ occupies a semicircle of radius b as shown in Fig. P4.10. Use the material presented in Example 4-4 to determine the electric field at the origin.

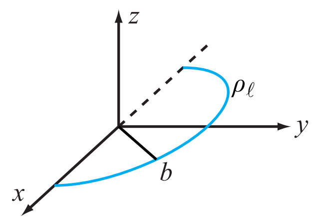

Figure P4.10: Problem 4.10.

Solution: Since we have only half of a circle, we need to integrate the expression for $d\mathbf{E}_1$ given in Example 4-4 over ϕ from 0 to π. Before we do that, however, we need to set $h = 0$ (the problem asks for $\mathbf{E}$ at the origin). Hence,

$$d\mathbf{E}_1 = \frac{\rho_l b}{4\pi\varepsilon_0} \frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2 + h^2)^{3/2}}\, d\phi \bigg|_{h=0}$$

$$= \frac{-\hat{\mathbf{r}}\rho_l}{4\pi\varepsilon_0 b}\, d\phi$$

$$\mathbf{E}_1 = \int_{\phi=0}^{\pi} d\mathbf{E}_1 = -\frac{-\hat{\mathbf{r}}\rho_l}{4\varepsilon_0 b}\ .$$

Problem 4.11 A square with sides of 2 m has a charge of 40 μC at each of its four corners. Determine the electric field at a point 5 m above the center of the square.

Solution: The distance $|R|$ between any of the charges and point P is

$$|R| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27}.$$

$$\begin{aligned}
\mathbf{E} &= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}_1}{|\mathbf{R}|^3} + \frac{\mathbf{R}_2}{|\mathbf{R}|^3} + \frac{\mathbf{R}_3}{|\mathbf{R}|^3} + \frac{\mathbf{R}_4}{|\mathbf{R}|^3}\right] \\
&= \frac{Q}{4\pi\varepsilon_0}\left[\frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} + \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} + \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} + \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}}\right] \\
&= \hat{\mathbf{z}}\,\frac{5Q}{(27)^{3/2}\pi\varepsilon_0} = \hat{\mathbf{z}}\,\frac{5 \times 40\ \mu\text{C}}{(27)^{3/2}\pi\varepsilon_0} = \frac{1.42}{\pi\varepsilon_0} \times 10^{-6}\ (\text{V/m}) = \hat{\mathbf{z}}\,51.2\ (\text{kV/m}).
\end{aligned}$$

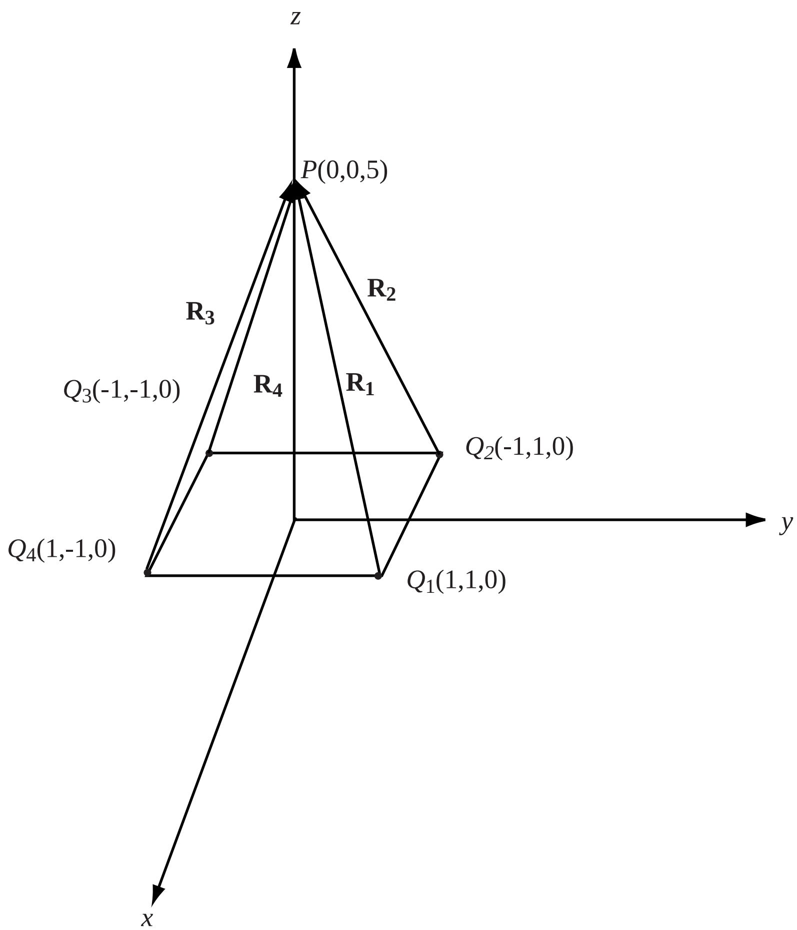

Figure P4.11: Square with charges at the corners.

Problem 4.22 Given the electric flux density

$$\mathbf{D} = \hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y) \quad (\text{C/m}^2)$$

determine

(a) ρ_v by applying Eq. (4.26).

(b) The total charge Q enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the x-, y-, and z-axes and one of its corners at the origin.

(c) The total charge Q in the cube, obtained by applying Eq. (4.29).

Solution:

(a) By applying Eq. (4.26)

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(3x-2y) = 0.$$

(b) Integrate the charge density over the volume as in Eq. (4.27):

$$Q = \int_{\mathcal{V}} \nabla \cdot \mathbf{D} \, d\mathcal{V} = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} 0 \, dx \, dy \, dz = 0.$$

(c) Apply Gauss' law to calculate the total charge from Eq. (4.29)

$$Q = \oint \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}},$$

$$F_{\text{front}} = \int_{y=0}^{2} \int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{x=2} \cdot (\hat{\mathbf{x}} \, dz \, dy)$$

$$= \int_{y=0}^{2} \int_{z=0}^{2} 2(x+y)\Big|_{x=2} \, dz \, dy = \left(2z \left(2y + \frac{1}{2}y^2\right) \Big|_{z=0}^{2} \right) \Big|_{y=0}^{2} = 24,$$

$$F_{\text{back}} = \int_{y=0}^{2} \int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{x=0} \cdot (-\hat{\mathbf{x}} \, dz \, dy)$$

$$= -\int_{y=0}^{2} \int_{z=0}^{2} 2(x+y)\Big|_{x=0} \, dz \, dy = -\left(zy^2 \Big|_{z=0}^{2} \right) \Big|_{y=0}^{2} = -8,$$

$$F_{\text{right}} = \int_{x=0}^{2} \int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{y=2} \cdot (\hat{\mathbf{y}} \, dz \, dx)$$

$$= \int_{x=0}^{2} \int_{z=0}^{2} (3x-2y)\Big|_{y=2} \, dz \, dx = \left(z \left(\frac{3}{2}x^2 - 4x\right) \Big|_{z=0}^{2} \right) \Big|_{x=0}^{2} = -4,$$

$$F_{\text{left}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\Big|_{y=0} \cdot (-\hat{\mathbf{y}}\, dz\, dx)$$

$$= -\int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\Big|_{y=0} dz\, dx = -\left(z\left(\frac{3}{2}x^2\right)\Big|_{z=0}^{2}\right)\Big|_{x=0}^{2} = -12,$$

$$F_{\text{top}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\Big|_{z=2} \cdot (\hat{\mathbf{z}}\, dy\, dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=2} dy\, dx = 0,$$

$$F_{\text{bottom}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\Big|_{z=0} \cdot (\hat{\mathbf{z}}\, dy\, dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=0} dy\, dx = 0.$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 24 - 8 - 4 - 12 + 0 + 0 = 0.$

Problem 4.25 The electric flux density inside a dielectric sphere of radius a centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}}\rho_0 R \qquad (\text{C/m}^2)$$

where ρ_0 is a constant. Find the total charge inside the sphere.

Solution:

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \hat{\mathbf{R}}\rho_0 R \cdot \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi \bigg|_{R=a}$$
$$= 2\pi\rho_0 a^3 \int_0^{\pi} \sin\theta \, d\theta = -2\pi\rho_0 a^3 \cos\theta \big|_0^{\pi} = 4\pi\rho_0 a^3 \quad (\text{C}).$$

Problem 4.28 If the charge density increases linearly with distance from the origin such that $\rho_v = 0$ at the origin and $\rho_v = 4$ C/m^3 at $R = 2$ m, find the corresponding variation of $\mathbf{D}$.

Solution:

$$\rho_v(R) = a + bR,$$
$$\rho_v(0) = a = 0,$$
$$\rho_v(2) = 2b = 40.$$

Hence, $b = 20$.

$$\rho_v(R) = 20R \quad (\text{C/m}^3).$$

Applying Gauss's law to a spherical surface of radius R,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho_v \, d\mathcal{V},$$
$$D_R \cdot 4\pi R^2 = \int_0^R 20R \cdot 4\pi R^2 \, dR = 80\pi \frac{R^4}{4},$$
$$D_R = 5R^2 \quad (\text{C/m}^2),$$
$$\mathbf{D} = \hat{\mathbf{R}} D_R = \hat{\mathbf{R}} 5R^2 \quad (\text{C/m}^2).$$

Problem 4.30 A square in the x–y plane in free space has a point charge of $+Q$ at corner $(a/2, a/2)$, the same at corner $(a/2, -a/2)$, and a point charge of $-Q$ at each of the other two corners.

(a) Find the electric potential at any point P along the x-axis.

(b) Evaluate V at $x = a/2$.

Solution: $R_1 = R_2$ and $R_3 = R_4$.

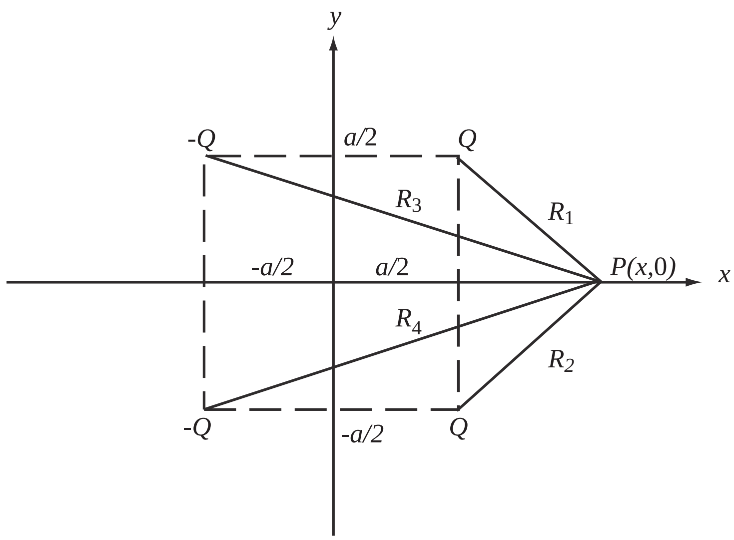

Figure P4.30: Potential due to four point charges.

$$V = \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q}{4\pi\varepsilon_0 R_2} + \frac{-Q}{4\pi\varepsilon_0 R_3} + \frac{-Q}{4\pi\varepsilon_0 R_4} = \frac{Q}{2\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_3}\right)$$

with

$$R_1 = \sqrt{\left(x - \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2},$$

$$R_3 = \sqrt{\left(x + \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}.$$

At $x = a/2$,

$$R_1 = \frac{a}{2},$$

$$R_3 = \frac{a\sqrt{5}}{2},$$

$$V = \frac{Q}{2\pi\varepsilon_0}\left(\frac{2}{a} - \frac{2}{\sqrt{5}\,a}\right) = \frac{0.55Q}{\pi\varepsilon_0 a}.$$

Problem 4.33 Show that the electric potential difference V_{12} between two points in air at radial distances r_1 and r_2 from an infinite line of charge with density ρ_ℓ along the z-axis is $V_{12} = (\rho_\ell/2\pi\varepsilon_0)\ln(r_2/r_1)$.

Solution: From Eq. (4.33), the electric field due to an infinite line of charge is

$$\mathbf{E} = \hat{\mathbf{r}} E_{\mathrm{r}} = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\varepsilon_0 r} .$$

Hence, the potential difference is

$$V_{12} = -\int_{r_2}^{r_1} \mathbf{E} \cdot d\mathbf{l} = -\int_{r_2}^{r_1} \frac{\hat{\mathbf{r}}\rho_l}{2\pi\varepsilon_0 r} \cdot \hat{\mathbf{r}}\, dr = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{r_2}{r_1}\right) .$$

Problem 4.35 For the electric dipole shown in Fig. 4-13, $d = 1$ cm and $|\mathbf{E}| = 4$ (mV/m) at $R = 1$ m and $\theta = 0°$. Find $\mathbf{E}$ at $R = 2$ m and $\theta = 90°$.

Solution: For $R = 1$ m and $\theta = 0°$, $|\mathbf{E}| = 4$ mV/m, we can solve for q using Eq. (4.56):

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}(\hat{\mathbf{R}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta).$$

Hence,

$$|\mathbf{E}| = \left(\frac{qd}{4\pi\varepsilon_0}\right)2 = 4 \text{ mV/m} \quad \text{at } \theta = 0°,$$

$$q = \frac{10^{-3} \times 8\pi\varepsilon_0}{d} = \frac{10^{-3} \times 8\pi\varepsilon_0}{10^{-2}} = 0.8\pi\varepsilon_0 \quad \text{(C)}.$$

Again using Eq. (4.56) to find $\mathbf{E}$ at $R = 2$ m and $\theta = 90°$, we have

$$\mathbf{E} = \frac{0.8\pi\varepsilon_0 \times 10^{-2}}{4\pi\varepsilon_0 \times 2^3}(\hat{\mathbf{R}}(0) + \hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\theta}}\,\frac{1}{4} \quad \text{(mV/m)}.$$