Problem 3.36 Find the gradient of the following scalar functions:

(a) $T = 3/(x^2 + z^2)$,

(b) $V = xy^2z^4$,

(c) $U = z\cos\phi/(1 + r^2)$,

(d) $W = e^{-R}\sin\theta$,

(e) $S = 4x^2e^{-z} + y^3$,

(f) $N = r^2\cos^2\phi$,

(g) $M = R\cos\theta\sin\phi$.

Solution:

(a) From Eq. (3.72),

$$\nabla T = -\hat{\mathbf{x}}\frac{6x}{(x^2+z^2)^2} - \hat{\mathbf{z}}\frac{6z}{(x^2+z^2)^2}\,.$$

(b) From Eq. (3.72),

$$\nabla V = \hat{\mathbf{x}}y^2z^4 + \hat{\mathbf{y}}2xyz^4 + \hat{\mathbf{z}}4xy^2z^3.$$

(c) From Eq. (3.82),

$$\nabla U = -\hat{\mathbf{r}}\frac{2rz\cos\phi}{(1+r^2)^2} - \hat{\boldsymbol{\phi}}\frac{z\sin\phi}{r(1+r^2)} + \hat{\mathbf{z}}\frac{\cos\phi}{1+r^2}\,.$$

(d) From Eq. (3.83),

$$\nabla W = -\hat{\mathbf{R}}e^{-R}\sin\theta + \hat{\boldsymbol{\theta}}(e^{-R}/R)\cos\theta.$$

(e) From Eq. (3.72),

$$S = 4x^2e^{-z} + y^3,$$

$$\nabla S = \hat{\mathbf{x}}\frac{\partial S}{\partial x} + \hat{\mathbf{y}}\frac{\partial S}{\partial y} + \hat{\mathbf{z}}\frac{\partial S}{\partial z} = \hat{\mathbf{x}}8xe^{-z} + \hat{\mathbf{y}}3y^2 - \hat{\mathbf{z}}4x^2e^{-z}.$$

(f) From Eq. (3.82),

$$N = r^2\cos^2\phi,$$

$$\nabla N = \hat{\mathbf{r}}\frac{\partial N}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial N}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial N}{\partial z} = \hat{\mathbf{r}}2r\cos^2\phi - \hat{\boldsymbol{\phi}}2r\sin\phi\cos\phi.$$

(g) From Eq. (3.83),

$$M = R\cos\theta\sin\phi,$$

$$\nabla M = \hat{\mathbf{R}}\frac{\partial M}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial M}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial M}{\partial\phi} = \hat{\mathbf{R}}\cos\theta\sin\phi - \hat{\boldsymbol{\theta}}\sin\theta\sin\phi + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{\tan\theta}\,.$$

Problem 3.40 For the scalar function $V = xy^2 - z^2$, determine its directional derivative along the direction of vector $\mathbf{A} = (\hat{\mathbf{x}} - \hat{\mathbf{y}}z)$ and then evaluate it at $P = (1, -1, 4)$.

Solution: The directional derivative is given by Eq. (3.75) as $dV/dl = \nabla V \cdot \hat{\mathbf{a}}_l$, where the unit vector in the direction of $\mathbf{A}$ is given by Eq. (3.2):

$$\hat{\mathbf{a}}_l = \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}z}{\sqrt{1+z^2}}\,,$$

and the gradient of V in Cartesian coordinates is given by Eq. (3.72):

$$\nabla V = \hat{\mathbf{x}}y^2 + \hat{\mathbf{y}}2xy - \hat{\mathbf{z}}2z.$$

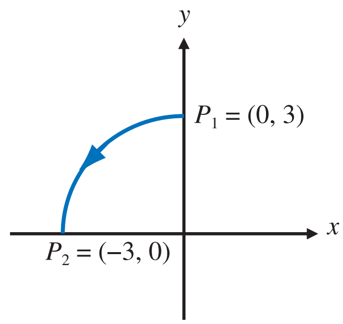
Therefore, by Eq. (3.75),

$$\frac{dV}{dl} = \frac{y^2 - 2xyz}{\sqrt{1+z^2}}\,.$$

At $P = (1, -1, 4)$,

$$\left(\frac{dV}{dl}\right)\bigg|_{(1,-1,4)} = \frac{9}{\sqrt{17}} = 2.18\,.$$

Problem 3.41 Evaluate the line integral of $\mathbf{E} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$ along the segment P_1 to P_2 of the circular path shown in the figure.



Solution: We need to calculate:

$$\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell}.$$

Since the path is along the perimeter of a circle, it is best to use cylindrical coordinates, which requires expressing both $\mathbf{E}$ and $d\boldsymbol{\ell}$ in cylindrical coordinates. Using Table 3-2,

$$\begin{aligned}\mathbf{E} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)r\cos\phi - (\hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)r\sin\phi \\ &= \hat{\mathbf{r}}\, r(\cos^2\phi - \sin^2\phi) - \hat{\boldsymbol{\phi}}2r\sin\phi\cos\phi\end{aligned}$$

The designated path is along the ϕ-direction at a constant $r = 3$. From Table 3-1, the applicable component of $d\boldsymbol{\ell}$ is:

$$d\boldsymbol{\ell} = \hat{\boldsymbol{\phi}}\, r\, d\phi.$$

Hence,

$$\begin{aligned}\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} &= \int_{\phi=90^\circ}^{\phi=180^\circ} \left[\hat{\mathbf{r}}\, r(\cos^2\phi - \sin^2\phi) - \hat{\boldsymbol{\phi}}\, 2r\sin\phi\cos\phi\right] \cdot \hat{\boldsymbol{\phi}}\, r\, d\phi \Big|_{r=3} \\ &= \int_{90^\circ}^{180^\circ} -2r^2 \sin\phi\cos\phi\; d\phi \Big|_{r=3} \\ &= -2r^2 \left.\left.\frac{\sin^2\phi}{2}\right|_{\phi=90^\circ}^{180^\circ}\right|_{r=3} = 9.\end{aligned}$$

Problem 3.46 For the vector field $\mathbf{E} = \hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy$, verify the divergence theorem by computing:

(a) the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes, and

(b) the integral of $\nabla \cdot \mathbf{E}$ over the cube's volume.

Solution:

(a) For a cube, the closed surface integral has 6 sides:

$$\oint \mathbf{E} \cdot d\mathbf{s} = F_{\text{top}} + F_{\text{bottom}} + F_{\text{right}} + F_{\text{left}} + F_{\text{front}} + F_{\text{back}},$$

$$F_{\text{top}} = \int_{x=-1}^{1} \int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{z=1} \cdot (\hat{\mathbf{z}}\, dy\, dx)$$

$$= -\int_{x=-1}^{1} \int_{y=-1}^{1} xy\, dy\, dx = \left(\left(\frac{x^2y^2}{4} \right) \bigg|_{y=-1}^{1} \right) \bigg|_{x=-1}^{1} = 0,$$

$$F_{\text{bottom}} = \int_{x=-1}^{1} \int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{z=-1} \cdot (-\hat{\mathbf{z}}\, dy\, dx)$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} xy\, dy\, dx = \left(\left(\frac{x^2y^2}{4} \right) \bigg|_{y=-1}^{1} \right) \bigg|_{x=-1}^{1} = 0,$$

$$F_{\text{right}} = \int_{x=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{y=1} \cdot (\hat{\mathbf{y}}\, dz\, dx)$$

$$= -\int_{x=-1}^{1} \int_{z=-1}^{1} z^2\, dz\, dx = -\left(\left(\frac{xz^3}{3} \right) \bigg|_{z=-1}^{1} \right) \bigg|_{x=-1}^{1} = \frac{-4}{3},$$

$$F_{\text{left}} = \int_{x=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{y=-1} \cdot (-\hat{\mathbf{y}}\, dz\, dx)$$

$$= -\int_{x=-1}^{1} \int_{z=-1}^{1} z^2\, dz\, dx = -\left(\left(\frac{xz^3}{3} \right) \bigg|_{z=-1}^{1} \right) \bigg|_{x=-1}^{1} = \frac{-4}{3},$$

$$F_{\text{front}} = \int_{y=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{x=1} \cdot (\hat{\mathbf{x}}\, dz\, dy)$$

$$= \int_{y=-1}^{1} \int_{z=-1}^{1} z\, dz\, dy = \left(\left(\frac{yz^2}{2} \right) \bigg|_{z=-1}^{1} \right) \bigg|_{y=-1}^{1} = 0,$$

$$F_{\text{back}} = \int_{y=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{x=-1} \cdot (-\hat{\mathbf{x}}\,dz\,dy)$$

$$= \int_{y=-1}^{1} \int_{z=-1}^{1} z\,dz\,dy = \left(\left(\frac{yz^2}{2} \right) \Bigg|_{z=-1}^{1} \right) \Bigg|_{y=-1}^{1} = 0,$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = 0 + 0 + \frac{-4}{3} + \frac{-4}{3} + 0 + 0 = \frac{-8}{3}\,.$$

(b)

$$\iiint \nabla \cdot \mathbf{E}\,dv = \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} \nabla \cdot (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy)\,dz\,dy\,dx$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} (z - z^2)\,dz\,dy\,dx$$

$$= \left(\left(\left(xy \left(\frac{z^2}{2} - \frac{z^3}{3} \right) \right) \Bigg|_{z=-1}^{1} \right) \Bigg|_{y=-1}^{1} \right) \Bigg|_{x=-1}^{1} = \frac{-8}{3}\,.$$

Problem 3.52 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B}\cdot d\mathbf{l}$ over the semicircular contour shown in Fig. P3.52(a), and

(b) $\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s}$ over the surface of the semicircle.

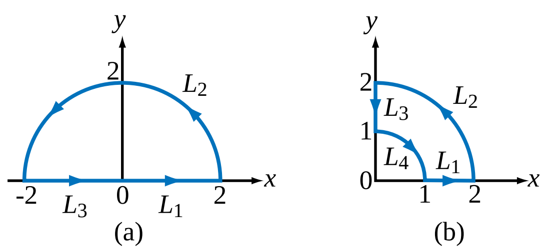


Figure P3.52: Contour paths for (a) Problem 3.52 and (b) Problem 3.53.

Solution:

(a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1} \mathbf{B}\cdot d\mathbf{l} + \int_{L_2} \mathbf{B}\cdot d\mathbf{l} + \int_{L_3} \mathbf{B}\cdot d\mathbf{l},$$

$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}} r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=0}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\;\; z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\big|_{r=0}^{2} + 0 = 2,$$

$$\int_{L_2} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{r=2,\;\; z=0}$$

$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi} = 4,$$

$$\int_{L_3} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{0} r\cos\phi\,dr\right)\bigg|_{\phi=\pi, z=0} + \left(\int_{\phi=\pi}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(-\tfrac{1}{2}r^2\right)\big|_{r=2}^{0} + 0 = 2,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = 2 + 4 + 2 = 8.$$

(b)

$$
\begin{aligned}
\nabla\times\mathbf{B} &= \nabla\times(\hat{\mathbf{r}}r\cos\phi+\hat{\boldsymbol{\phi}}\sin\phi)\\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0-\frac{\partial}{\partial z}(\sin\phi)\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi)-\frac{\partial}{\partial r}0\right)\\
&\quad+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi))-\frac{\partial}{\partial\phi}(r\cos\phi)\right)\\
&= \hat{\mathbf{r}}0+\hat{\boldsymbol{\phi}}0+\hat{\mathbf{z}}\frac{1}{r}(\sin\phi+(r\sin\phi))=\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right),\\
\iint\nabla\times\mathbf{B}\cdot d\mathbf{s} &= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)\\
&= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\sin\phi(r+1)\,dr\,d\phi=\left(\left(-\cos\phi\left(\tfrac{1}{2}r^2+r\right)\right)\Big|_{r=0}^{2}\right)\Big|_{\phi=0}^{\pi}=8.
\end{aligned}
$$

Problem 3.56 Determine if each of the following vector fields is solenoidal, conservative, or both:

(a) $\mathbf{A} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y2xy$,

(b) $\mathbf{B} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z$,

(c) $\mathbf{C} = \hat{\mathbf{r}}(\sin\phi)/r^2 + \hat{\boldsymbol{\phi}}(\cos\phi)/r^2$,

(d) $\mathbf{D} = \hat{\mathbf{R}}/R$,

(e) $\mathbf{E} = \hat{\mathbf{r}}\left(3 - \frac{r}{1+r}\right) + \hat{\mathbf{z}}z$,

(f) $\mathbf{F} = (\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)/(x^2+y^2)$,

(g) $\mathbf{G} = \hat{\mathbf{x}}(x^2+z^2) - \hat{\mathbf{y}}(y^2+x^2) - \hat{\mathbf{z}}(y^2+z^2)$,

(h) $\mathbf{H} = \hat{\mathbf{R}}(Re^{-R})$.

Solution:

(a)

$$\nabla\cdot\mathbf{A} = \nabla\cdot(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}2xy = 2x - 2x = 0,$$

$$\begin{aligned}\nabla\times\mathbf{A} &= \nabla\times(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy)\\ &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}(-2xy)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}0\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2)\right)\\ &= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 - \hat{\mathbf{z}}(2y) \neq 0.\end{aligned}$$

The field **A** is solenoidal but not conservative.

(b)

$$\nabla\cdot\mathbf{B} = \nabla\cdot(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}2z = 2x - 2y + 2 \neq 0,$$

$$\begin{aligned}\nabla\times\mathbf{B} &= \nabla\times(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z)\\ &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(-y^2)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(2z)\right)\\ &\quad + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(x^2)\right)\\ &= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 + \hat{\mathbf{z}}0.\end{aligned}$$

The field **B** is conservative but not solenoidal.

(c)

$$\begin{aligned}\nabla\cdot\mathbf{C} &= \nabla\cdot\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2} + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\sin\phi}{r^2}\right)\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{r^2}\right) + \frac{\partial}{\partial z}0\\ &= \frac{-\sin\phi}{r^3} + \frac{-\sin\phi}{r^3} + 0 = \frac{-2\sin\phi}{r^3},\end{aligned}$$

$$
\begin{aligned}
\nabla\times\mathbf{C} &= \nabla\times\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2}+\hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0-\frac{\partial}{\partial z}\left(\frac{\cos\phi}{r^2}\right)\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}\left(\frac{\sin\phi}{r^2}\right)-\frac{\partial}{\partial r}0\right)\\
&\quad+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\left(\frac{\cos\phi}{r^2}\right)\right)-\frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{r^2}\right)\right)\\
&= \hat{\mathbf{r}}0+\hat{\boldsymbol{\phi}}0+\hat{\mathbf{z}}\frac{1}{r}\left(-\left(\frac{\cos\phi}{r^2}\right)-\left(\frac{\cos\phi}{r^2}\right)\right)=\hat{\mathbf{z}}\frac{-2\cos\phi}{r^3}.
\end{aligned}
$$

The field **C** is neither solenoidal nor conservative.

(d)

$$
\nabla\cdot\mathbf{D}=\nabla\cdot\left(\frac{\hat{\mathbf{R}}}{R}\right)=\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\left(\frac{1}{R}\right)\right)+\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(0\sin\theta)+\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}0=\frac{1}{R^2},
$$

$$
\begin{aligned}
\nabla\times\mathbf{D} &= \nabla\times\left(\frac{\hat{\mathbf{R}}}{R}\right)\\
&= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(0\sin\theta)-\frac{\partial}{\partial\phi}0\right)+\hat{\boldsymbol{\theta}}\frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right)-\frac{\partial}{\partial R}(R(0))\right)\\
&\quad+\hat{\boldsymbol{\phi}}\frac{1}{R}\left(\frac{\partial}{\partial R}(R(0))-\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\right)=\hat{\mathbf{r}}0+\hat{\boldsymbol{\theta}}0+\hat{\boldsymbol{\phi}}0.
\end{aligned}
$$

The field **D** is conservative but not solenoidal.

(e)

$$
\begin{aligned}
\mathbf{E} &= \hat{\mathbf{r}}\left(3-\frac{r}{1+r}\right)+\hat{\mathbf{z}}z,\\
\nabla\cdot\mathbf{E} &= \frac{1}{r}\frac{\partial}{\partial r}(rE_r)+\frac{1}{r}\frac{\partial E_\phi}{\partial\phi}+\frac{\partial E_z}{\partial z}\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(3r-\frac{r^2}{1+r}\right)+1\\
&= \frac{1}{r}\left[3-\frac{2r}{1+r}+\frac{r^2}{(1+r)^2}\right]+1\\
&= \frac{1}{r}\left[\frac{3+3r^2+6r-2r-2r^2+r^2}{(1+r)^2}\right]+1=\frac{2r^2+4r+3}{r(1+r)^2}+1\neq 0,\\
\nabla\times\mathbf{E} &= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial E_z}{\partial\phi}-\frac{\partial E_\phi}{\partial z}\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}\right)+\hat{\mathbf{z}}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)-\frac{1}{r}\frac{\partial E_r}{\partial\phi}\right)=0.
\end{aligned}
$$

Hence, **E** is conservative, but not solenoidal.

(f)

$$\mathbf{F} = \frac{\hat{\mathbf{x}}y + \hat{\mathbf{y}}x}{x^2+y^2} = \hat{\mathbf{x}}\,\frac{y}{x^2+y^2} + \hat{\mathbf{y}}\,\frac{x}{x^2+y^2}\,,$$

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{-2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} \neq 0,$$

$$\nabla\times\mathbf{F} = \hat{\mathbf{x}}(0-0) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]$$

$$= \hat{\mathbf{z}}\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right)$$

$$= \hat{\mathbf{z}}\,\frac{2(y^2-x^2)}{(x^2+y^2)^2} \neq 0.$$

Hence, **F** is neither solenoidal nor conservative.

(g)

$$\mathbf{G} = \hat{\mathbf{x}}(x^2+z^2) - \hat{\mathbf{y}}(y^2+x^2) - \hat{\mathbf{z}}(y^2+z^2),$$

$$\nabla\cdot\mathbf{G} = \frac{\partial}{\partial x}(x^2+z^2) - \frac{\partial}{\partial y}(y^2+x^2) - \frac{\partial}{\partial z}(y^2+z^2)$$

$$= 2x - 2y - 2z \neq 0,$$

$$\nabla\times\mathbf{G} = \hat{\mathbf{x}}\left(-\frac{\partial}{\partial y}(y^2+z^2) + \frac{\partial}{\partial z}(y^2+x^2)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2+z^2) + \frac{\partial}{\partial x}(y^2+z^2)\right)$$

$$+\hat{\mathbf{z}}\left(-\frac{\partial}{\partial x}(y^2+x^2) - \frac{\partial}{\partial y}(x^2+z^2)\right)$$

$$= -\hat{\mathbf{x}}\,2y + \hat{\mathbf{y}}\,2z - \hat{\mathbf{z}}\,2x \neq 0.$$

Hence, **G** is neither solenoidal nor conservative.

(h)

$$\mathbf{H} = \hat{\mathbf{R}}(Re^{-R}),$$

$$\nabla\cdot\mathbf{H} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^3e^{-R}) = \frac{1}{R^2}(3R^2e^{-R} - R^3e^{-R}) = e^{-R}(3-R) \neq 0,$$

$$\nabla\times\mathbf{H} = 0.$$

Hence, **H** is conservative, but not solenoidal.

Problem 3.57 Find the Laplacian of the following scalar functions:

(a) $V = 4xy^2z^3$,
(b) $V = xy + yz + zx$,
(c) $V = 3/(x^2 + y^2)$,
(d) $V = 5e^{-r}\cos\phi$,
(e) $V = 10e^{-R}\sin\theta$.

Solution:

(a) From Eq. (3.110), $\nabla^2(4xy^2z^3) = 8xz^3 + 24xy^2z$.
(b) $\nabla^2(xy + yz + zx) = 0$.
(c) From the inside back cover of the book,

$$\nabla^2\left(\frac{3}{x^2+y^2}\right) = \nabla^2(3r^{-2}) = 12r^{-4} = \frac{12}{(x^2+y^2)^2}.$$

(d)

$$\nabla^2(5e^{-r}\cos\phi) = 5e^{-r}\cos\phi\left(1 - \frac{1}{r} - \frac{1}{r^2}\right).$$

(e)

$$\nabla^2(10e^{-R}\sin\theta) = 10e^{-R}\left[\sin\theta\left(1 - \frac{2}{R}\right) + \frac{\cos^2\theta - \sin^2\theta}{R^2\sin\theta}\right].$$