Problem 3.1 Vector **A** starts at point $(1,-1,-3)$ and ends at point $(2,-1,0)$. Find a unit vector in the direction of **A**.

Solution:

$$\begin{aligned}
\mathbf{A} &= \hat{\mathbf{x}}(2-1)+\hat{\mathbf{y}}(-1-(-1))+\hat{\mathbf{z}}(0-(-3)) = \hat{\mathbf{x}}+\hat{\mathbf{z}}3, \\
|\mathbf{A}| &= \sqrt{1+9} = 3.16, \\
\hat{\mathbf{a}} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}}+\hat{\mathbf{z}}3}{3.16} = \hat{\mathbf{x}}0.32+\hat{\mathbf{z}}0.95.
\end{aligned}$$

Problem 3.2 Given vectors $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}$, $\mathbf{B} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3$, and $\mathbf{C} = \hat{\mathbf{x}}4 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2$, show that $\mathbf{C}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

Solution:

$$\mathbf{A} \cdot \mathbf{C} = (\hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}) \cdot (\hat{\mathbf{x}}4 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2) = 8 - 6 - 2 = 0,$$

$$\mathbf{B} \cdot \mathbf{C} = (\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) \cdot (\hat{\mathbf{x}}4 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2) = 8 - 2 - 6 = 0.$$

Problem 3.5 Given vectors $\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3$, $\mathbf{B} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}4$, and $\mathbf{C} = \hat{\mathbf{y}}2 - \hat{\mathbf{z}}4$, find

(a) A and $\hat{\mathbf{a}}$,

(b) the component of $\mathbf{B}$ along $\mathbf{C}$,

(c) θ_{AC},

(d) $\mathbf{A} \times \mathbf{C}$,

(e) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$,

(f) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$,

(g) $\hat{\mathbf{x}} \times \mathbf{B}$, and

(h) $(\mathbf{A} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}}$.

Solution:

(a) From Eq. (3.4),

$$A = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14},$$

and, from Eq. (3.5),

$$\hat{\mathbf{a}}_A = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3}{\sqrt{14}}\,.$$

(b) The component of $\mathbf{B}$ along $\mathbf{C}$ (see Section 3-1.4) is given by

$$B\cos\theta_{BC} = \frac{\mathbf{B} \cdot \mathbf{C}}{C} = \frac{-8}{\sqrt{20}} = -1.8\,.$$

(c) From Eq. (3.18),

$$\theta_{AC} = \cos^{-1}\frac{\mathbf{A} \cdot \mathbf{C}}{AC} = \cos^{-1}\frac{4+12}{\sqrt{14}\sqrt{20}} = \cos^{-1}\frac{16}{\sqrt{280}} = 17.0^\circ.$$

(d) From Eq. (3.27),

$$\mathbf{A} \times \mathbf{C} = \hat{\mathbf{x}}(2(-4) - (-3)2) + \hat{\mathbf{y}}((-3)0 - 1(-4)) + \hat{\mathbf{z}}(1(2) - 2(0)) = -\hat{\mathbf{x}}2 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}2.$$

(e) From Eq. (3.27) and Eq. (3.21),

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot (\hat{\mathbf{x}}16 + \hat{\mathbf{y}}8 + \hat{\mathbf{z}}4) = 1(16) + 2(8) + (-3)4 = 20.$$

Eq. (3.30) could also have been used in the solution. Also, Eq. (3.29) could be used in conjunction with the result of part (d).

(f) By repeated application of Eq. (3.27),

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \times (\hat{\mathbf{x}}16 + \hat{\mathbf{y}}8 + \hat{\mathbf{z}}4) = \hat{\mathbf{x}}32 - \hat{\mathbf{y}}52 - \hat{\mathbf{z}}24.$$

Eq. (3.33) could also have been used.

(g) From Eq. (3.27),

$$\hat{\mathbf{x}} \times \mathbf{B} = -\hat{\mathbf{z}}4.$$

(h) From Eq. (3.27) and Eq. (3.21),

$$(\mathbf{A} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}} = (\hat{\mathbf{x}}3 + \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = 1.$$

Eq. (3.29) and Eq. (3.25) could also have been used in the solution.

Problem 3.11 Find a unit vector parallel to either direction of the line described by

$$2x + z = 4.$$

Solution: First, we find any two points on the given line. Since the line equation is not a function of y, the given line is in a plane parallel to the x–z plane. For convenience, we choose the x–z plane with $y = 0$.

For $x = 0$, $z = 4$. Hence, point P is at $(0, 0, 4)$.

For $z = 0$, $x = 2$. Hence, point Q is at $(2, 0, 0)$.

Vector $\mathbf{A}$ from P to Q is:

$$\mathbf{A} = \hat{\mathbf{x}}(2 - 0) + \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(0 - 4) = \hat{\mathbf{x}}2 - \hat{\mathbf{z}}4,$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}}2 - \hat{\mathbf{z}}4}{\sqrt{20}}\,.$$

Problem 3.16 Given $\mathbf{B} = \hat{\mathbf{x}}(z-3y) + \hat{\mathbf{y}}(2x-3z) - \hat{\mathbf{z}}(x+y)$, find a unit vector parallel to $\mathbf{B}$ at point $P = (1, 0, -1)$.

Solution: At $P = (1, 0, -1)$,

$$\mathbf{B} = \hat{\mathbf{x}}(-1) + \hat{\mathbf{y}}(2+3) - \hat{\mathbf{z}}(1) = -\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}},$$

$$\hat{\mathbf{b}} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}}{\sqrt{1+25+1}} = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}}{\sqrt{27}}.$$

Problem 3.22 Convert the coordinates of the following points from Cartesian to cylindrical and spherical coordinates:

(a) $P_1 = (1,2,0)$,

(b) $P_2 = (0,0,2)$,

(c) $P_3 = (1,1,3)$,

(d) $P_4 = (-2,2,-2)$.

Solution: Use the "coordinate variables" column in Table 3-2.

(a) In the cylindrical coordinate system,

$$P_1 = (\sqrt{1^2+2^2}, \tan^{-1}(2/1), 0) = (\sqrt{5}, 1.107 \text{ rad}, 0) \approx (2.24, 63.4^\circ, 0).$$

In the spherical coordinate system,

$$\begin{aligned} P_1 &= (\sqrt{1^2+2^2+0^2}, \tan^{-1}(\sqrt{1^2+2^2}/0), \tan^{-1}(2/1)) \\ &= (\sqrt{5}, \pi/2 \text{ rad}, 1.107 \text{ rad}) \approx (2.24, 90.0^\circ, 63.4^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant I.

(b) In the cylindrical coordinate system,

$$P_2 = (\sqrt{0^2+0^2}, \tan^{-1}(0/0), 2) = (0, 0 \text{ rad}, 2) = (0, 0^\circ, 2).$$

In the spherical coordinate system,

$$\begin{aligned} P_2 &= (\sqrt{0^2+0^2+2^2}, \tan^{-1}(\sqrt{0^2+0^2}/2), \tan^{-1}(0/0)) \\ &= (2, 0 \text{ rad}, 0 \text{ rad}) = (2, 0^\circ, 0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is arbitrary and may take any value.

(c) In the cylindrical coordinate system,

$$P_3 = (\sqrt{1^2+1^2}, \tan^{-1}(1/1), 3) = (\sqrt{2}, \pi/4 \text{ rad}, 3) \approx (1.41, 45.0^\circ, 3).$$

In the spherical coordinate system,

$$\begin{aligned} P_3 &= (\sqrt{1^2+1^2+3^2}, \tan^{-1}(\sqrt{1^2+1^2}/3), \tan^{-1}(1/1)) \\ &= (\sqrt{11}, 0.44 \text{ rad}, \pi/4 \text{ rad}) \approx (3.32, 25.2^\circ, 45.0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant I.

(d) In the cylindrical coordinate system,

$$\begin{aligned} P_4 &= (\sqrt{(-2)^2+2^2}, \tan^{-1}(2/-2), -2) \\ &= (2\sqrt{2}, 3\pi/4 \text{ rad}, -2) \approx (2.83, 135.0^\circ, -2). \end{aligned}$$

In the spherical coordinate system,

$$\begin{aligned}P_4 &= (\sqrt{(-2)^2+2^2+(-2)^2}, \tan^{-1}(\sqrt{(-2)^2+2^2}/-2), \tan^{-1}(2/-2)) \\ &= (2\sqrt{3}, 2.187 \text{ rad}, 3\pi/4 \text{ rad}) \approx (3.46, 125.3^\circ, 135.0^\circ).\end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant II.

Problem 3.25 Use the appropriate expression for the differential surface area $d\mathbf{s}$ to determine the area of each of the following surfaces:

(a) $r = 3;\ 0 \le \phi \le \pi/3;\ -2 \le z \le 2,$

(b) $2 \le r \le 5;\ \pi/2 \le \phi \le \pi;\ z = 0,$

(c) $2 \le r \le 5;\ \phi = \pi/4;\ -2 \le z \le 2,$

(d) $R = 2;\ 0 \le \theta \le \pi/3;\ 0 \le \phi \le \pi,$

(e) $0 \le R \le 5;\ \theta = \pi/3;\ 0 \le \phi \le 2\pi.$

Also sketch the outlines of each of the surfaces.

Solution:

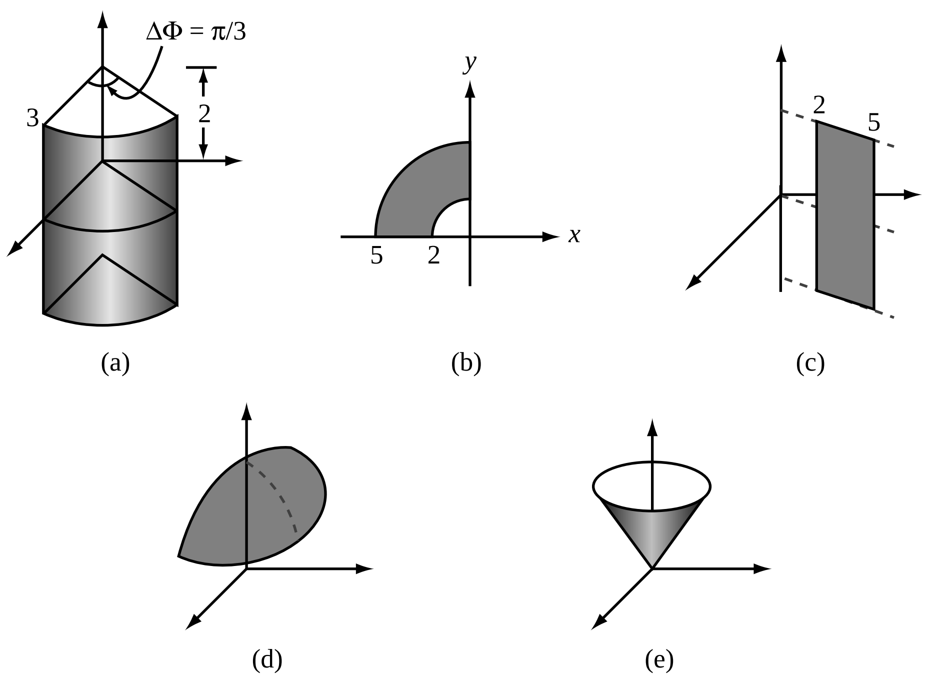

Figure P3.25: Surfaces described by Problem 3.25.

(a) Using Eq. (3.43a),

$$A = \int_{z=-2}^{2} \int_{\phi=0}^{\pi/3} (r)|_{r=3}\, d\phi\, dz = \left((3\phi z)\big|_{\phi=0}^{\pi/3} \right)\Big|_{z=-2}^{2} = 4\pi.$$

(b) Using Eq. (3.43c),

$$A = \int_{r=2}^{5} \int_{\phi=\pi/2}^{\pi} (r)|_{z=0}\, d\phi\, dr = \left(\left(\tfrac{1}{2} r^2 \phi\right)\big|_{r=2}^{5} \right)\Big|_{\phi=\pi/2}^{\pi} = \frac{21\pi}{4}\,.$$

(c) Using Eq. (3.43b),

$$A = \int_{z=-2}^{2} \int_{r=2}^{5} (1)\big|_{\phi=\pi/4}\, dr\, dz = \left((rz)\big|_{z=-2}^{2} \right)\Big|_{r=2}^{5} = 12.$$

(d) Using Eq. (3.50b),

$$A = \int_{\theta=0}^{\pi/3} \int_{\phi=0}^{\pi} \left(R^2 \sin\theta\right)\Big|_{R=2}\, d\phi\, d\theta = \left((-4\phi\cos\theta)\big|_{\theta=0}^{\pi/3} \right)\Big|_{\phi=0}^{\pi} = 2\pi.$$

(e) Using Eq. (3.50c),

$$A = \int_{R=0}^{5} \int_{\phi=0}^{2\pi} (R\sin\theta)\big|_{\theta=\pi/3}\, d\phi\, dR = \left(\left(\tfrac{1}{2} R^2 \phi \sin\frac{\pi}{3} \right)\Big|_{\phi=0}^{2\pi} \right)\Bigg|_{R=0}^{5} = \frac{25\sqrt{3}\pi}{2}.$$

Problem 3.26 Find the volumes described by

(a) $2 \le r \le 5;\ \pi/2 \le \phi \le \pi;\ 0 \le z \le 2,$

(b) $0 \le R \le 5;\ 0 \le \theta \le \pi/3;\ 0 \le \phi \le 2\pi.$

Also sketch the outline of each volume.

Solution:

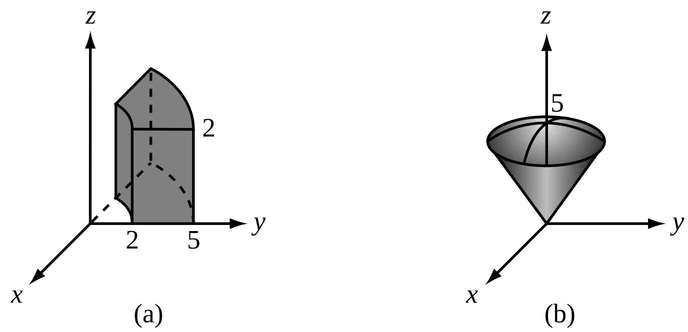

Figure P3.26: Volumes described by Problem 3.26 .

(a) From Eq. (3.44),

$$\mathcal{V} = \int_{z=0}^{2} \int_{\phi=\pi/2}^{\pi} \int_{r=2}^{5} r\,dr\,d\phi\,dz = \left(\left(\left(\tfrac{1}{2} r^2 \phi z\right)\Big|_{r=2}^{5} \right) \Big|_{\phi=\pi/2}^{\pi} \right) \Big|_{z=0}^{2} = \frac{21\pi}{2}\,.$$

(b) From Eq. (3.50e),

$$\begin{aligned} \mathcal{V} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/3} \int_{R=0}^{5} R^2 \sin\theta\,dR\,d\theta\,d\phi \\ &= \left(\left(\left(-\cos\theta \frac{R^3}{3} \phi \right) \Big|_{R=0}^{5} \right) \Big|_{\theta=0}^{\pi/3} \right) \Big|_{\phi=0}^{2\pi} = \frac{125\pi}{3}\,. \end{aligned}$$

Problem 3.30 Given vectors

$$\mathbf{A} = \hat{\mathbf{r}}(\cos\phi + 3z) - \hat{\boldsymbol{\phi}}(2r + 4\sin\phi) + \hat{\mathbf{z}}(r - 2z),$$
$$\mathbf{B} = -\hat{\mathbf{r}}\sin\phi + \hat{\mathbf{z}}\cos\phi,$$

find

(a) θ_{AB} at $(2, \pi/2, 0)$,

(b) a unit vector perpendicular to both **A** and **B** at $(2, \pi/3, 1)$.

Solution: It doesn't matter whether the vectors are evaluated before vector products are calculated, or if the vector products are directly calculated and the general results are evaluated at the specific point in question.

(a) At $(2, \pi/2, 0)$, $\mathbf{A} = -\hat{\boldsymbol{\phi}}8 + \hat{\mathbf{z}}2$ and $\mathbf{B} = -\hat{\mathbf{r}}$. From Eq. (3.18),

$$\theta_{AB} = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) = \cos^{-1}\left(\frac{0}{AB}\right) = 90^\circ.$$

(b) At $(2, \pi/3, 1)$, $\mathbf{A} = \hat{\mathbf{r}}\frac{7}{2} - \hat{\boldsymbol{\phi}}4(1 + \frac{1}{2}\sqrt{3})$ and $\mathbf{B} = -\hat{\mathbf{r}}\frac{1}{2}\sqrt{3} + \hat{\mathbf{z}}\frac{1}{2}$. Since $\mathbf{A}\times\mathbf{B}$ is perpendicular to both **A** and **B**, a unit vector perpendicular to both **A** and **B** is given by

$$\begin{aligned}\pm\frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} &= \pm\frac{\hat{\mathbf{r}}(-4(1+\frac{1}{2}\sqrt{3}))(\frac{1}{2}) - \hat{\boldsymbol{\phi}}(\frac{7}{2})(\frac{1}{2}) - \hat{\mathbf{z}}(4(1+\frac{1}{2}\sqrt{3}))(\frac{1}{2}\sqrt{3})}{\sqrt{(2(1+\frac{1}{2}\sqrt{3}))^2 + (\frac{7}{4})^2 + (3+2\sqrt{3})^2}} \\ &\approx \mp(\hat{\mathbf{r}}0.487 + \hat{\boldsymbol{\phi}}0.228 + \hat{\mathbf{z}}0.843).\end{aligned}$$

Problem 3.32 Determine the distance between the following pairs of points:

(a) $P_1 = (1,1,2)$ and $P_2 = (0,2,3)$,

(b) $P_3 = (2,\pi/3,1)$ and $P_4 = (4,\pi/2,3)$,

(c) $P_5 = (3,\pi,\pi/2)$ and $P_6 = (4,\pi/2,\pi)$.

Solution:

(a) From Eq. (3.66),

$$d = \sqrt{(0-1)^2 + (2-1)^2 + (3-2)^2} = \sqrt{3}.$$

(b) From Eq. (3.67),

$$d = \sqrt{2^2 + 4^2 - 2(2)(4)\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + (3-1)^2} = \sqrt{24 - 8\sqrt{3}} \approx 3.18.$$

(c) From Eq. (3.68),

$$d = \sqrt{3^2 + 4^2 - 2(3)(4)\left(\cos\frac{\pi}{2}\cos\pi + \sin\pi\sin\frac{\pi}{2}\cos\left(\pi - \frac{\pi}{2}\right)\right)} = 5.$$

Problem 3.34 Transform the following vectors into cylindrical coordinates and then evaluate them at the indicated points:

(a) $\mathbf{A} = \hat{\mathbf{x}}(x+y)$ at $P_1 = (1,2,3)$,

(b) $\mathbf{B} = \hat{\mathbf{x}}(y-x) + \hat{\mathbf{y}}(x-y)$ at $P_2 = (1,0,2)$,

(c) $\mathbf{C} = \hat{\mathbf{x}}y^2/(x^2+y^2) - \hat{\mathbf{y}}x^2/(x^2+y^2) + \hat{\mathbf{z}}4$ at $P_3 = (1,-1,2)$,

(d) $\mathbf{D} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta + \hat{\boldsymbol{\phi}}\cos^2\phi$ at $P_4(2,\pi/2,\pi/4)$,

(e) $\mathbf{E} = \hat{\mathbf{R}}\cos\phi + \hat{\boldsymbol{\theta}}\sin\phi + \hat{\boldsymbol{\phi}}\sin^2\theta$ at $P_5 = (3,\pi/2,\pi)$.

Solution: From Table 3-2:

(a)

$$\begin{aligned}
\mathbf{A} &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(r\cos\phi + r\sin\phi) \\
&= \hat{\mathbf{r}}r\cos\phi(\cos\phi+\sin\phi) - \hat{\boldsymbol{\phi}}r\sin\phi(\cos\phi+\sin\phi), \\
P_1 &= (\sqrt{1^2+2^2}, \tan^{-1}(2/1), 3) = (\sqrt{5}, 63.4^\circ, 3), \\
\mathbf{A}(P_1) &= (\hat{\mathbf{r}}0.447 - \hat{\boldsymbol{\phi}}0.894)\sqrt{5}(.447+.894) = \hat{\mathbf{r}}1.34 - \hat{\boldsymbol{\phi}}2.68.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{B} &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(r\sin\phi - r\cos\phi) + (\hat{\boldsymbol{\phi}}\cos\phi + \hat{\mathbf{r}}\sin\phi)(r\cos\phi - r\sin\phi) \\
&= \hat{\mathbf{r}}r(2\sin\phi\cos\phi - 1) + \hat{\boldsymbol{\phi}}r(\cos^2\phi - \sin^2\phi) = \hat{\mathbf{r}}r(\sin 2\phi - 1) + \hat{\boldsymbol{\phi}}r\cos 2\phi, \\
P_2 &= (\sqrt{1^2+0^2}, \tan^{-1}(0/1), 2) = (1, 0^\circ, 2), \\
\mathbf{B}(P_2) &= -\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{C} &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)\frac{r^2\sin^2\phi}{r^2} - (\hat{\boldsymbol{\phi}}\cos\phi + \hat{\mathbf{r}}\sin\phi)\frac{r^2\cos^2\phi}{r^2} + \hat{\mathbf{z}}4 \\
&= \hat{\mathbf{r}}\sin\phi\cos\phi(\sin\phi - \cos\phi) - \hat{\boldsymbol{\phi}}(\sin^3\phi + \cos^3\phi) + \hat{\mathbf{z}}4, \\
P_3 &= (\sqrt{1^2+(-1)^2}, \tan^{-1}(-1/1), 2) = (\sqrt{2}, -45^\circ, 2), \\
\mathbf{C}(P_3) &= \hat{\mathbf{r}}0.707 + \hat{\mathbf{z}}4.
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{D} &= (\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\sin\theta + (\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\cos\theta + \hat{\boldsymbol{\phi}}\cos^2\phi = \hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}\cos^2\phi, \\
P_4 &= (2\sin(\pi/2), \pi/4, 2\cos(\pi/2)) = (2, 45^\circ, 0), \\
\mathbf{D}(P_4) &= \hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}\tfrac{1}{2}.
\end{aligned}$$

(e)

$$\mathbf{E} = (\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\cos\phi + (\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\sin\phi + \hat{\boldsymbol{\phi}}\sin^2\theta,$$

$$P_5 = \left(3, \frac{\pi}{2}, \pi\right),$$

$$\mathbf{E}(P_5) = \left(\hat{\mathbf{r}}\sin\frac{\pi}{2} + \hat{\mathbf{z}}\cos\frac{\pi}{2}\right)\cos\pi + \left(\hat{\mathbf{r}}\cos\frac{\pi}{2} - \hat{\mathbf{z}}\sin\frac{\pi}{2}\right)\sin\pi + \hat{\boldsymbol{\phi}}\sin^2\frac{\pi}{2} = -\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}.$$

Problem 3.35 Transform the following vectors into spherical coordinates and then evaluate them at the indicated points:

(a) $\mathbf{A} = \hat{\mathbf{x}}y^2 + \hat{\mathbf{y}}xz + \hat{\mathbf{z}}4$ at $P_1 = (1,-1,2)$,

(b) $\mathbf{B} = \hat{\mathbf{y}}(x^2+y^2+z^2) - \hat{\mathbf{z}}(x^2+y^2)$ at $P_2 = (-1,0,2)$,

(c) $\mathbf{C} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi + \hat{\mathbf{z}}\cos\phi\sin\phi$ at $P_3 = (2,\pi/4,2)$, and

(d) $\mathbf{D} = \hat{\mathbf{x}}y^2/(x^2+y^2) - \hat{\mathbf{y}}x^2/(x^2+y^2) + \hat{\mathbf{z}}4$ at $P_4 = (1,-1,2)$.

Solution: From Table 3-2:

(a)

$$\begin{aligned}
\mathbf{A} &= (\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(R\sin\theta\sin\phi)^2 \\
&\quad + (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)(R\sin\theta\cos\phi)(R\cos\theta) \\
&\quad + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)4 \\
&= \hat{\mathbf{R}}(R^2\sin^2\theta\sin\phi\cos\phi(\sin\theta\sin\phi + \cos\theta) + 4\cos\theta) \\
&\quad + \hat{\boldsymbol{\theta}}(R^2\sin\theta\cos\theta\sin\phi\cos\phi(\sin\theta\sin\phi + \cos\theta) - 4\sin\theta) \\
&\quad + \hat{\boldsymbol{\phi}}R^2\sin\theta(\cos\theta\cos^2\phi - \sin\theta\sin^3\phi), \\
P_1 &= \left(\sqrt{1^2+(-1)^2+2^2}, \tan^{-1}\left(\sqrt{1^2+(-1)^2}/2\right), \tan^{-1}(-1/1)\right) \\
&= (\sqrt{6}, 35.3^\circ, -45^\circ), \\
\mathbf{A}(P_1) &\approx \hat{\mathbf{R}}2.856 - \hat{\boldsymbol{\theta}}2.888 + \hat{\boldsymbol{\phi}}2.123.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{B} &= (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)R^2 - (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)R^2\sin^2\theta \\
&= \hat{\mathbf{R}}R^2\sin\theta(\sin\phi - \sin\theta\cos\theta) + \hat{\boldsymbol{\theta}}R^2(\cos\theta\sin\phi + \sin^3\theta) + \hat{\boldsymbol{\phi}}R^2\cos\phi, \\
P_2 &= \left(\sqrt{(-1)^2+0^2+2^2}, \tan^{-1}\left(\sqrt{(-1)^2+0^2}/2\right), \tan^{-1}(0/(-1))\right) \\
&= (\sqrt{5}, 26.6^\circ, 180^\circ), \\
\mathbf{B}(P_2) &\approx -\hat{\mathbf{R}}0.896 + \hat{\boldsymbol{\theta}}0.449 - \hat{\boldsymbol{\phi}}5.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{C} &= (\hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta)\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\cos\phi\sin\phi \\
&= \hat{\mathbf{R}}\cos\phi(\sin\theta + \cos\theta\sin\phi) + \hat{\boldsymbol{\theta}}\cos\phi(\cos\theta - \sin\theta\sin\phi) - \hat{\boldsymbol{\phi}}\sin\phi, \\
P_3 &= \left(\sqrt{2^2+2^2}, \tan^{-1}(2/2), \pi/4\right) = (2\sqrt{2}, 45^\circ, 45^\circ), \\
\mathbf{C}(P_3) &\approx \hat{\mathbf{R}}0.854 + \hat{\boldsymbol{\theta}}0.146 - \hat{\boldsymbol{\phi}}0.707.
\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{D} &= (\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)\,\frac{R^2\sin^2\theta\sin^2\phi}{R^2\sin^2\theta\sin^2\phi + R^2\sin^2\theta\cos^2\phi}\\
&\quad - (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)\,\frac{R^2\sin^2\theta\cos^2\phi}{R^2\sin^2\theta\sin^2\phi + R^2\sin^2\theta\cos^2\phi}\\
&\quad + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)4\\
&= \hat{\mathbf{R}}(\sin\theta\cos\phi\sin^2\phi - \sin\theta\sin\phi\cos^2\phi + 4\cos\theta)\\
&\quad + \hat{\boldsymbol{\theta}}(\cos\theta\cos\phi\sin^2\phi - \cos\theta\sin\phi\cos^2\phi - 4\sin\theta)\\
&\quad - \hat{\boldsymbol{\phi}}(\cos^3\phi + \sin^3\phi),\end{aligned}$$

$$\begin{aligned}P_4(1,-1,2) &= P_4\left[\sqrt{1+1+4}, \tan^{-1}(\sqrt{1+1}/2), \tan^{-1}(-1/1)\right]\\
&= P_4(\sqrt{6}, 35.26^\circ, -45^\circ),\end{aligned}$$

$$\begin{aligned}\mathbf{D}(P_4) &= \hat{\mathbf{R}}(\sin 35.26^\circ\cos 45^\circ\sin^2 45^\circ - \sin 35.26^\circ\sin(-45^\circ)\cos^2 45^\circ + 4\cos 35.26^\circ)\\
&\quad + \hat{\boldsymbol{\theta}}(\cos 35.26^\circ\cos 45^\circ\sin^2 45^\circ - \cos 35.26^\circ\sin(-45^\circ)\cos^2 45^\circ - 4\sin 35.26^\circ)\\
&\quad - \hat{\boldsymbol{\phi}}(\cos^3 45^\circ + \sin^3 45^\circ)\\
&= \hat{\mathbf{R}}3.67 - \hat{\boldsymbol{\theta}}1.73 - \hat{\boldsymbol{\phi}}0.707.\end{aligned}$$