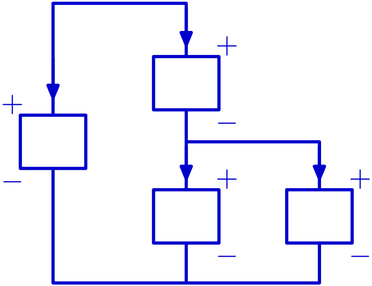
The Circuit Abstraction

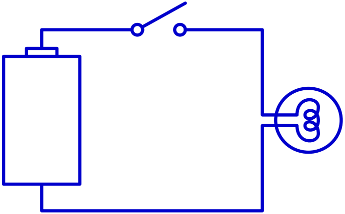
Circuits represent systems as connections of elements

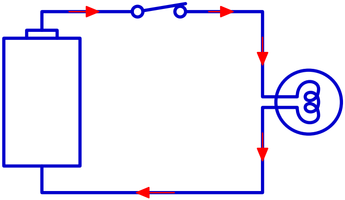
- through which currents (through variables) flow and
- across which voltages (across variables) develop.



The Circuit Abstraction

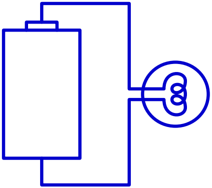
Current flows through a flashlight when the switch is closed

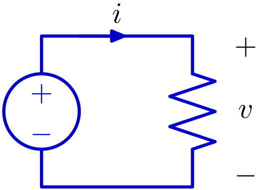




The Circuit Abstraction

We can represent the flashlight as a voltage source (battery) connected to a resistor (light bulb).

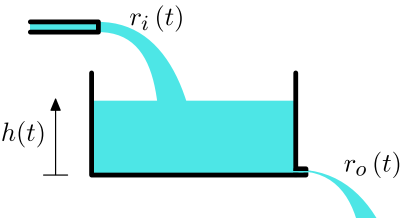


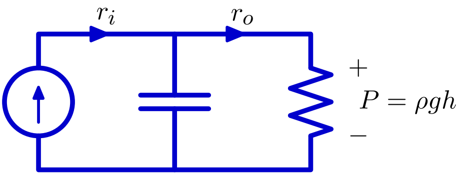


The voltage source generates a voltage v across the resistor and a current i through the resistor.

The Circuit Abstraction

We can represent the flow of water by a circuit.





Flow of water into and out of tank are represented as "through" variables r_i and r_o, respectively. Hydraulic pressure at bottom of tank is represented by the "across" variable $P = \rho g h$.

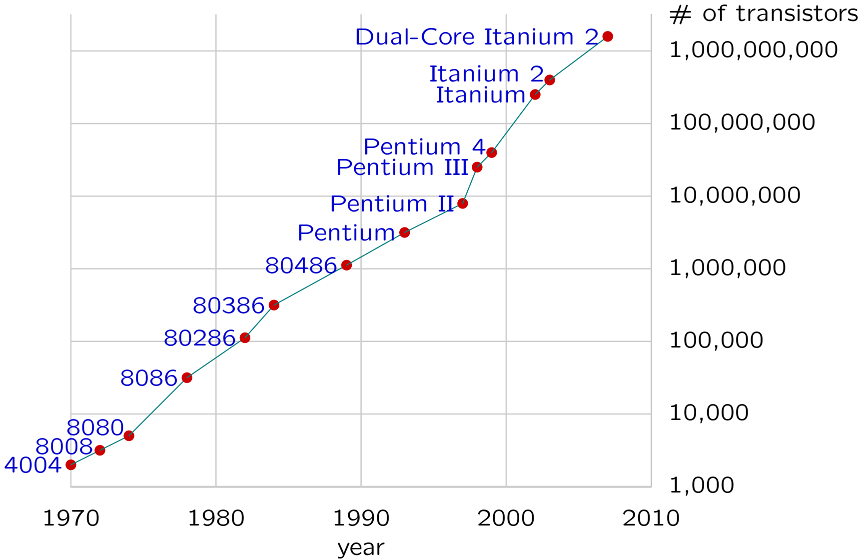
The Circuit Abstraction

Circuits are important for two very different reasons:

- as **physical systems**
  - power (from generators and transformers to power lines)
  - electronics (from cell phones to computers)
- as **models** of complex systems
  - neurons
  - brain
  - cardiovascular system
  - hearing

The Circuit Abstraction

Circuits are basis of enormously successful semiconductor industry.

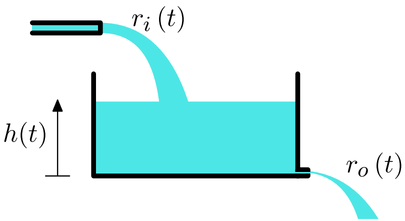


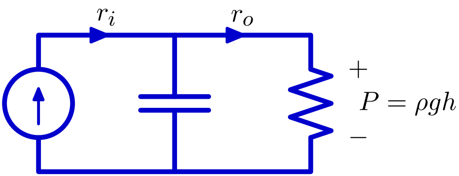
What design principles enable development of such complex systems?

The Circuit Abstraction

Circuits represent systems as connections of elements

- through which currents (through variables) flow and
- across which voltages (across variables) develop.





The **primitives** are the elements:

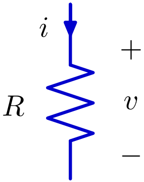
- sources,
- capacitors, and
- resistors.

The **rules of combination** are the rules that govern

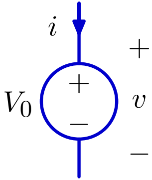
- flow of current (through variable) and
- development of voltage (across variable).

Analyzing Circuits: Elements

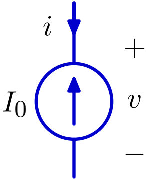
We will start with the simplest elements: resistors and sources



$v = iR$



$v = V_0$

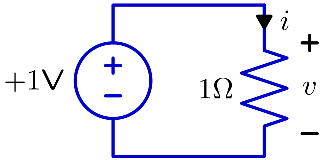


$i = -I_0$

Analyzing Simple Circuits

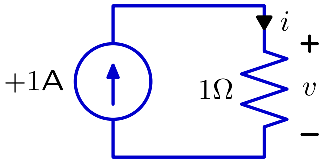
Analyzing simple circuits is straightforward.

Example 1:



The voltage source determines the voltage across the resistor, $v =$ 1V, so the current through the resistor is $i = v/R = 1/1 = 1$A.

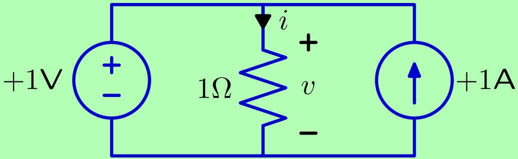
Example 2:



The current source determines the current through the resistor, $i =$ 1A, so the voltage across the resistor is $v = iR = 1 \times 1 = 1$V.

Check Yourself

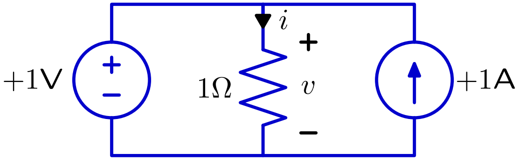
What is the current through the resistor below?



1. 1A
2. 2A
3. 0A
4. cannot determine
5. none of the above

Check Yourself

What is the current through the resistor below?

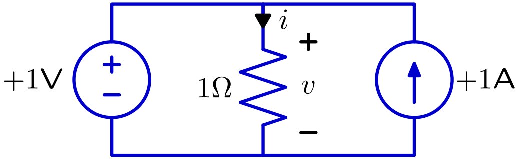


The voltage source forces the voltage across the resistor to be 1V. Therefore, the current through the resistor is $1V/1\Omega = 1A$.

Does the current source do anything?

Check Yourself

Does the current source do anything?



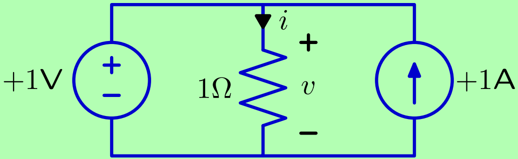
If all of the current from current source flowed through the resistor, then it would generate 1V across the resistor.

Since the voltage generated by the current source is equal to that across the voltage source, the voltage source provides zero current.

The current source supplies all of the current through the resistor!

Check Yourself

What is the current through the resistor below?



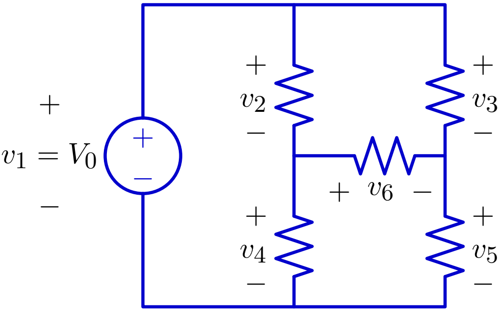
1. 1A
2. 2A
3. 0A
4. cannot determine
5. none of the above

Analyzing More Complex Circuits

More complex circuits can be analyzed by systematically applying Kirchhoff's voltage law (KVL) and Kirchhoff's current law (KCL).

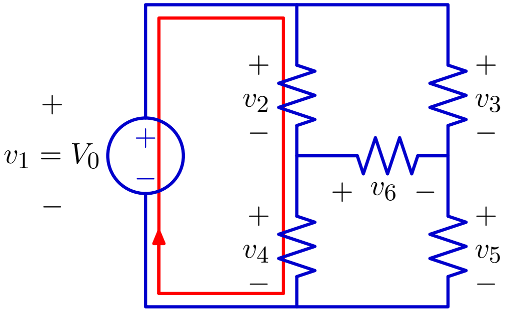
Analyzing Circuits: KVL

KVL: The sum of the voltages around any closed path is zero.



Analyzing Circuits: KVL

KVL: The sum of the voltages around any closed path is zero.

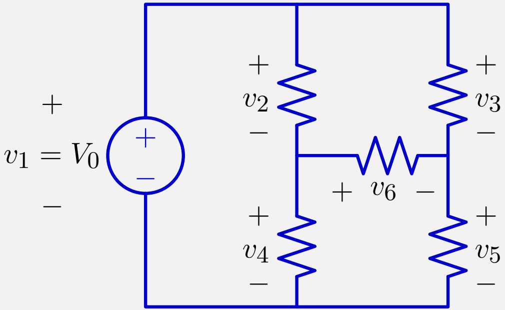


Example: $-v_1 + v_2 + v_4 = 0$ or equivalently $v_1 = v_2 + v_4$.

How many other KVL relations are there?

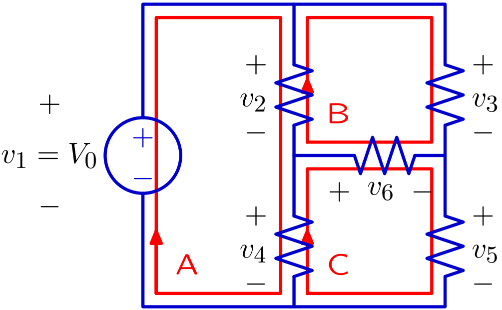
Check Yourself

How many KVL equations can be written for this circuit?



1. 3
2. 4
3. 5
4. 6
5. 7

Check Yourself

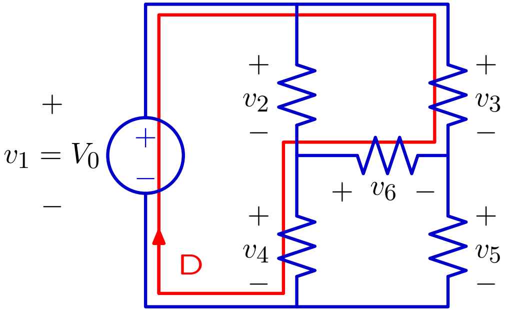


$$A: -v_1 + v_2 + v_4 = 0$$

$$B: -v_2 + v_3 - v_6 = 0$$

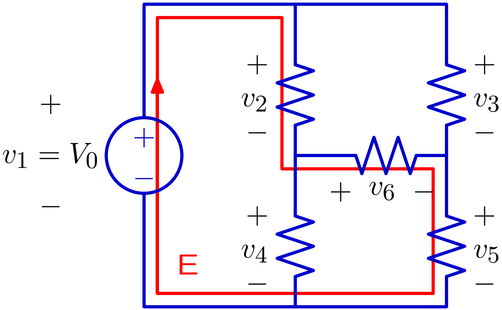
$$C: -v_4 + v_6 + v_5 = 0$$

Check Yourself



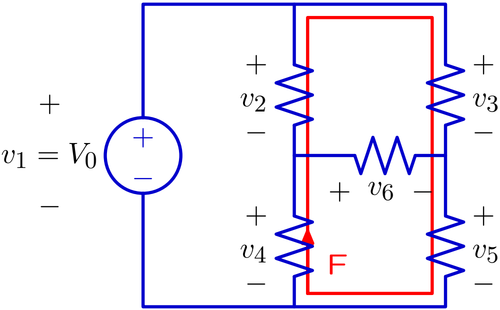
$$\text{D} : -v_1 + v_3 - v_6 + v_4 = 0$$

Check Yourself



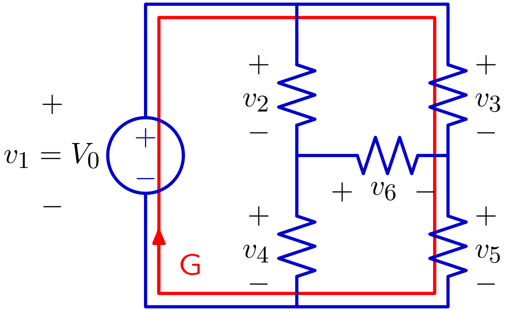
E : $-v_1 + v_2 + v_6 + v_5 = 0$

Check Yourself



$\mathsf{F}: -v_4 - v_2 + v_3 + v_5 = 0$

Check Yourself



$$\mathsf{G} : -v_1 + v_3 + v_5 = 0$$

Check Yourself

There are 7 KVL equations for this circuit.

$$\text{A}: -v_1 + v_2 + v_4 = 0$$

$$\text{B}: -v_2 + v_3 - v_6 = 0$$

$$\text{C}: -v_4 + v_6 + v_5 = 0$$

$$\text{D}: -v_1 + v_3 - v_6 + v_4 = 0$$

$$\text{E}: -v_1 + v_2 + v_6 + v_5 = 0$$

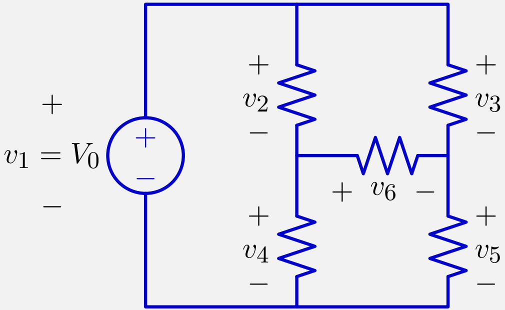
$$\text{F}: -v_4 - v_2 + v_3 + v_5 = 0$$

$$\text{G}: -v_1 + v_3 + v_5 = 0$$

Not all of these equations are linearly independent.

Check Yourself

How many KVL equations can be written for this circuit?



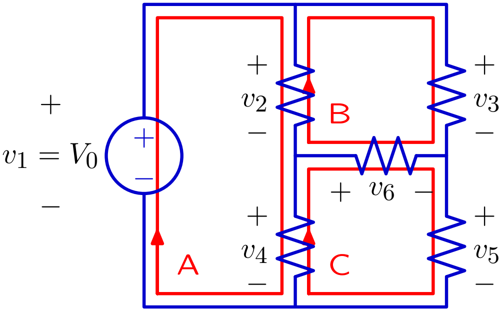
1. 3
2. 4
3. 5
4. 6
5. 7

But not all of these equations are linearly independent.

Analyzing Circuits: KVL

Planar circuits can be characterized by their "inner" loops.

KVL equations for the inner loops are independent.



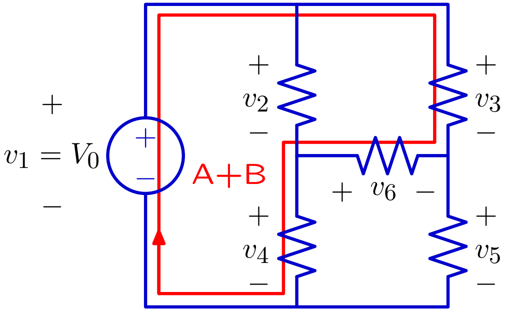
A : $-v_1 + v_2 + v_4 = 0$

B : $-v_2 + v_3 - v_6 = 0$

C : $-v_4 + v_6 + v_5 = 0$

Analyzing Circuits: KVL

All possible KVL equations for planar circuits can be generated by combinations of the "inner" loops.



$$\text{A}: -v_1 + v_2 + v_4 = 0$$

$$\text{B}: -v_2 + v_3 - v_6 = 0$$

$$\text{A+B}: -v_1 + v_2 + v_4 - v_2 + v_3 - v_6 = -v_1 + v_3 - v_6 + v_4 = 0$$

KVL: Summary

The sum of the voltages around any closed path is zero.

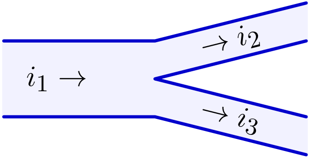
One KVL equation can be written for every closed path in a circuit.

Sets of KVL equations are not necessarily linearly independent.

KCL equations for the “inner” loops of planar circuits are linearly independent.

Kirchhoff's Current Law

The flow of electrical current is analogous to the flow of incompressible fluid (e.g., water).

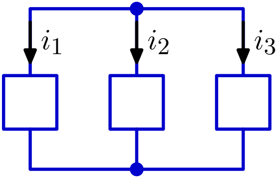


Current i_1 flows into a **node** and two currents i_2 and i_3 flow out:

$$i_1 = i_2 + i_3$$

Kirchhoff's Current Law

The net flow of electrical current into (or out of) a **node** is zero.



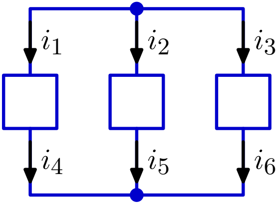
Here, there are two nodes, each indicated by a dot.

The net current out of the top node must be zero:

$$i_1 + i_2 + i_3 = 0\,.$$

Kirchhoff's Current Law

Electrical currents cannot accumulate in elements, so current that flows into a circuit element must also flow out.



$$i_1 = i_4$$

$$i_2 = i_5$$

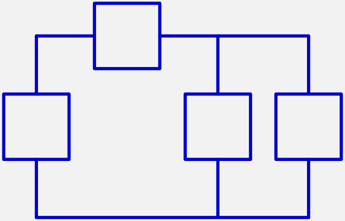
$$i_3 = i_6$$

Since $i_1 + i_2 + i_3 = 0$ it follows that

$$i_4 + i_5 + i_6 = 0\,.$$

Check Yourself

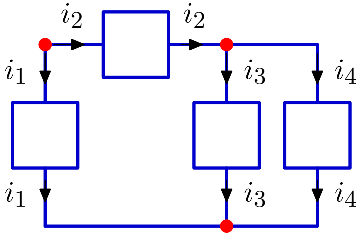
How many linearly independent KCL equations can be written for the following circuit?



1. 1
2. 2
3. 3
4. 4
5. 5

Check Yourself

How many linearly independent KCL equations can be written for the following circuit?



There are four **element currents**: i_1, i_2, i_3, and i_4.

We can write a KCL equation at each of the three **nodes**:

$$i_1 + i_2 = 0$$

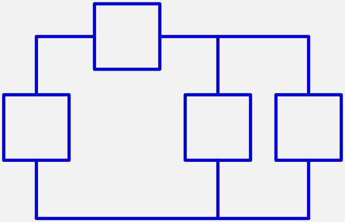
$$i_2 = i_3 + i_4$$

$$i_1 + i_3 + i_4 = 0$$

Substituting i_2 from the second equation into the first yields the third equation. Only two of these equations are linearly independent.

Check Yourself

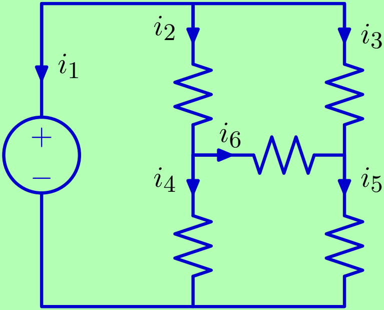
How many linearly independent KCL equations can be written for the following circuit? 2



1. 1
2. 2
3. 3
4. 4
5. 5

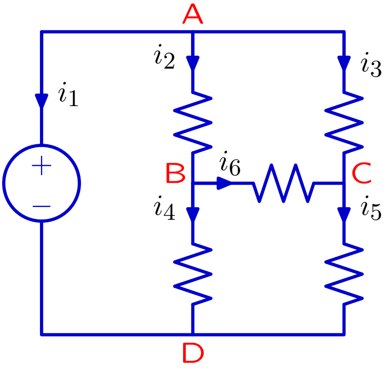
Check Yourself

How many distinct KCL relations can be written for this circuit?



1. 3
2. 4
3. 5
4. 6
5. 7

Check Yourself



A : $i_1 + i_2 + i_3 = 0$

B : $-i_2 + i_4 + i_6 = 0$

C : $-i_6 - i_3 + i_5 = 0$

D : $i_1 + i_4 + i_5 = 0$

Check Yourself

These equations are not linearly independent.

$$
\begin{aligned}
1: &\quad i_1 + i_2 + i_3 = 0 \\
2: &\quad -i_2 + i_4 + i_6 = 0 \\
3: &\quad -i_6 - i_3 + i_5 = 0 \\
4: &\quad i_1 + i_4 + i_5 = 0
\end{aligned}
$$

Substitute i_2 from 2 and i_3 from 3 into 1.

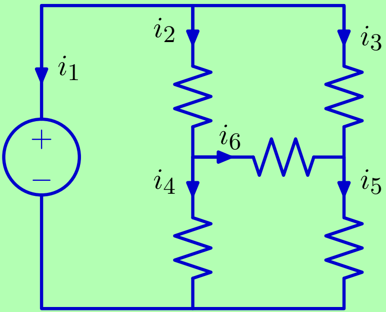
$$i_1 + (i_4 + i_6) + (i_5 - i_6) = i_1 + i_4 + i_5$$

This is equation 4!

There are only 3 linearly independent KCL equations.

Check Yourself

How many distinct KCL relations can be written for this circuit?

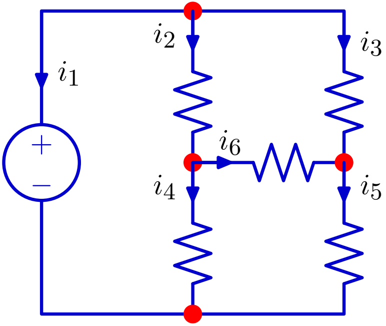


1. 3
2. 4
3. 5
4. 6
5. 7

Analyzing Circuits: KCL

The number of independent KCL equations is one less than the number of nodes.

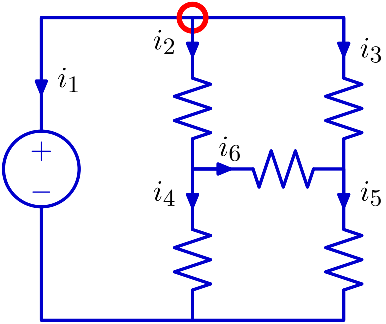
Previous circuit: four nodes and three independent KCL equations.



This relation follows from a generalization of KCL, as follows.

Analyzing Circuits: KCL

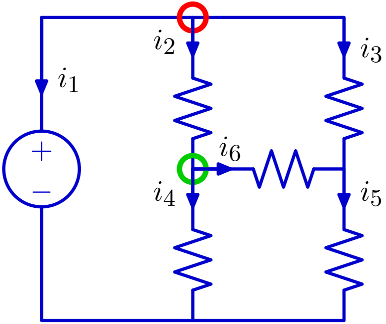
The net current out of any closed surface (which can contain multiple nodes) is zero.



node 1: $i_1 + i_2 + i_3 = 0$

Analyzing Circuits: KCL

The net current out of any closed surface (which can contain multiple nodes) is zero.

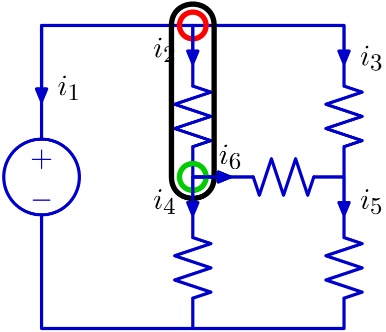


node 1: $i_1 + i_2 + i_3 = 0$

node 2: $-i_2 + i_4 + i_6 = 0$

Analyzing Circuits: KCL

The net current out of any closed surface (which can contain multiple nodes) is zero.



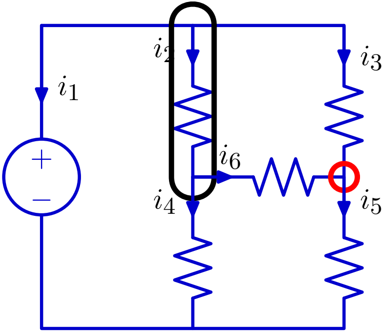
node 1: $i_1 + i_2 + i_3 = 0$

node 2: $-i_2 + i_4 + i_6 = 0$

nodes 1+2: $i_1 + i_2 + i_3 - i_2 + i_4 + i_6 = i_1 + i_3 + i_4 + i_6 = 0$

Analyzing Circuits: KCL

The net current out of any closed surface (which can contain multiple nodes) is zero.

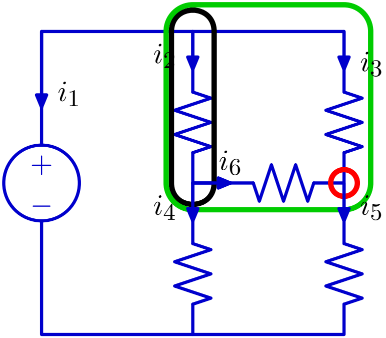


nodes 1+2: $i_1 + i_2 + i_3 - i_2 + i_4 + i_6 = i_1 + i_3 + i_4 + i_6 = 0$

node 3: $-i_3 - i_6 + i_5 = 0$

Analyzing Circuits: KCL

The net current out of any closed surface (which can contain multiple nodes) is zero.



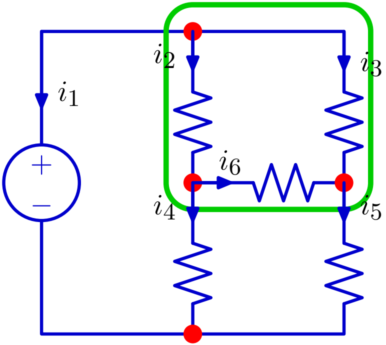
nodes 1+2: $i_1 + i_3 + i_4 + i_6 = 0$

node 3: $-i_3 - i_6 + i_5 = 0$

nodes 1+2+3: $i_1 + i_3 + i_4 + i_6 - i_3 - i_6 + i_5 = i_1 + i_4 + i_5 = 0$

Analyzing Circuits: KCL

The net current out of any closed surface (which can contain multiple nodes) is zero.



nodes 1+2: $i_1 + i_3 + i_4 + i_6 = 0$

node 3: $-i_3 - i_6 + i_5 = 0$

nodes 1+2+3: $i_1 + i_3 + i_4 + i_6 - i_3 - i_6 + i_5 = i_1 + i_4 + i_5 = 0$

Net current out of nodes 1+2+3 = net current into bottom node!

KCL: Summary

The sum of the currents out of any node is zero.

One KCL equation can be written for every closed surface (which contain one or more nodes) in a circuit.

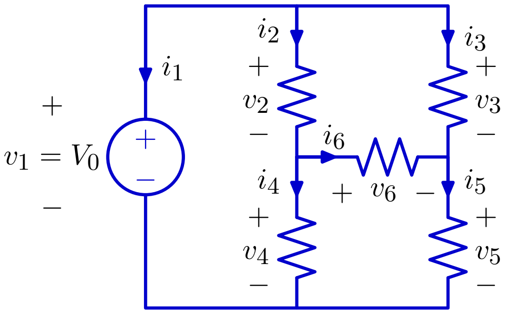
Sets of KCL equations are not necessarily linearly independent.

KCL equations for every primitive node except one (ground) are linearly independent.

KVL, KCL, and Constitutive Equations

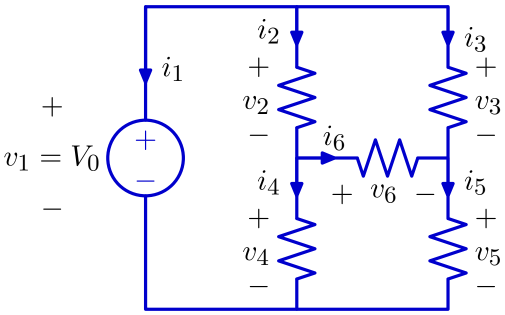
Circuits can be analyzed by combining

- all linearly independent KVL equations,
- all linearly independent KCL equations, and
- one constitutive equation for each element.



KVL, KCL, and Constitutive Equations

Unfortunately, there are a lot of equations and unknowns.



12 unknowns: v_1, v_2, v_3, v_4, v_5, v_6, i_1, i_2, i_3, i_4, i_5 and i_6.

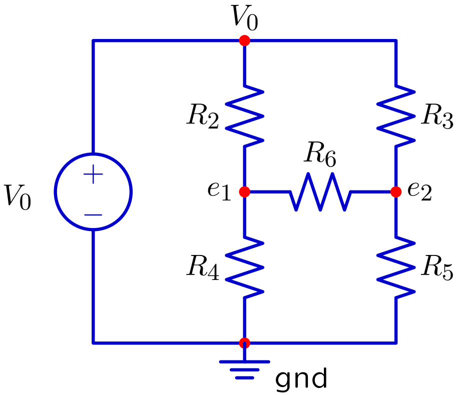
12 equations: 3 KVL + 3 KCL + 5 for resistors + 1 for V source

This circuit is characterized by 12 equations in 12 unknowns!

Node Voltages

The "node" method is one (of many) ways to systematically reduce the number of circuit equations and unknowns.

- label all nodes except one: ground (gnd) $\equiv$ 0 volts
- write KCL for each node whose voltage is not known



KCL at e_1:

$$\frac{e_1 - V_0}{R_2} + \frac{e_1 - e_2}{R_6} + \frac{e_1}{R_4} = 0$$

KCL at e_2:

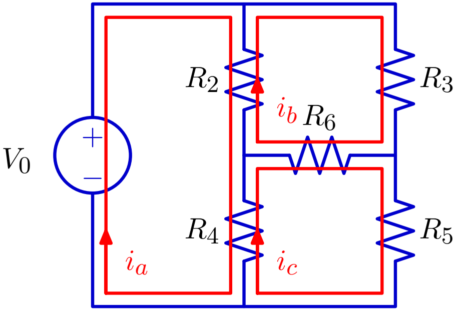
$$\frac{e_2 - V_0}{R_3} + \frac{e_2 - e_1}{R_6} + \frac{e_2}{R_5} = 0$$

- solve (here just 2 equations and 2 unknowns)

Loop Currents

The "loop current" method is another way to systematically reduce the number of circuit equations and unknowns.

- label all the loop currents
- write KVL for each loop



loop a:

$$-V_0 + R_2(i_a - i_b) + R_4(i_a - i_c) = 0$$

loop b:

$$R_2(i_b - i_a) + R_3(i_b) + R_6(i_b - i_c) = 0$$

loop c:

$$R_4(i_c - i_a) + +R_6(i_c - i_b) + R_5(i_c) = 0$$

- solve (here just 3 equations and 3 unknowns)

Analyzing Circuits: Summary

We have seen three (of many) methods for **analyzing** circuits.

Each one is based on a different set of variables:

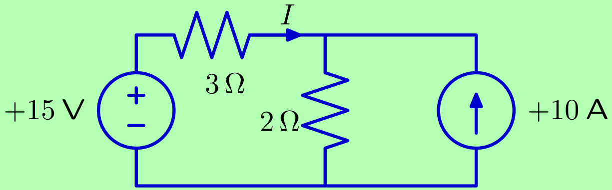
- currents and voltages for each element
- node voltages
- loop currents

Each requires the use of all constitutive equations.

Each provides a systematic way of identifying the required set of KVL and/or KCL equations.

Check Yourself

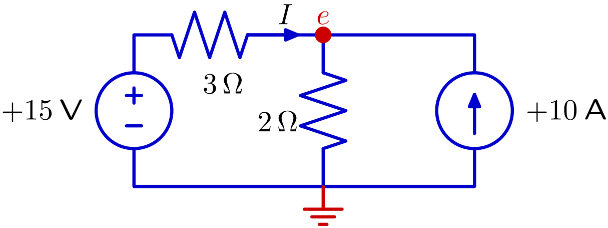
Determine the current I in the circuit below.



1. 1 A
2. $\frac{5}{3}$ A
3. -1 A
4. -5 A
5. none of the above

Check Yourself

Node method:



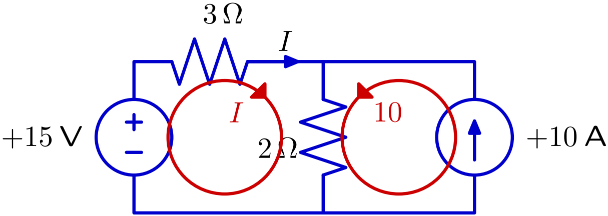
KCL at node e:

$$\frac{e-15}{3}+\frac{e}{2}=10 \quad \rightarrow \quad \frac{5}{6}e=15 \quad \rightarrow \quad e=18$$

$$I=\frac{15-18}{3}=-1\,\text{A}$$

Check Yourself

Loop method:

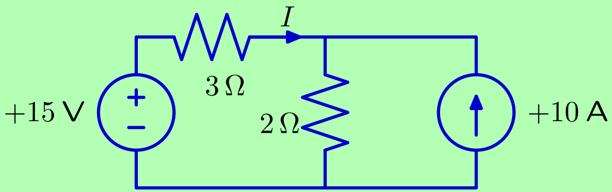


KVL for left loop:

$$-15 + 3I + 2(I + 10) = 0 \quad \rightarrow \quad 5I = -5 \quad \rightarrow \quad I = -1\text{ A}$$

Check Yourself

Determine the current I in the circuit below. 3

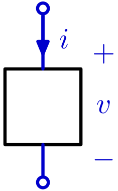


1. 1 A
2. $\frac{5}{3}$ A
3. -1 A
4. -5 A
5. none of the above

Common Patterns

Circuits can be simplified when two or more elements behave as a single element.

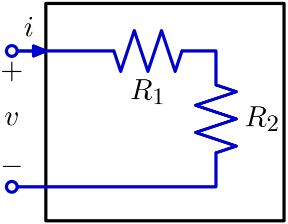
A "one-port" is a circuit that can be represented as a single element.

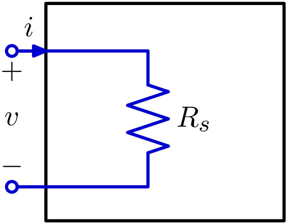


A one-port has two terminals. Current enters one terminal (+) and exits the other (−), producing a voltage (v) across the terminals.

Series Combinations

The series combination of two resistors is equivalent to a single resistor whose resistance is the sum of the two original resistances.





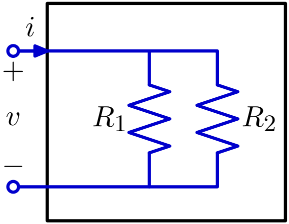
$$v = R_1 i + R_2 i \qquad v = R_s i$$

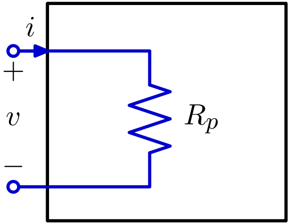
$$R_s = R_1 + R_2$$

The resistance of a series combination is always **larger** than either of the original resistances.

Parallel Combinations

The parallel combination of two resistors is equivalent to a single resistor whose conductance (1/resistance) is the sum of the two original conductances.





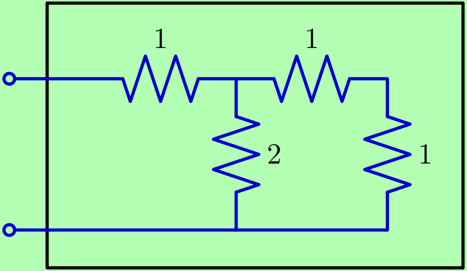
$$i = \frac{v}{R_1} + \frac{v}{R_2} \qquad\qquad i = \frac{v}{R_p}$$

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2} \quad \rightarrow \quad R_p = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2} \equiv R_1 || R_2$$

The resistance of a parallel combination is always **smaller** than either of the original resistances.

Check Yourself

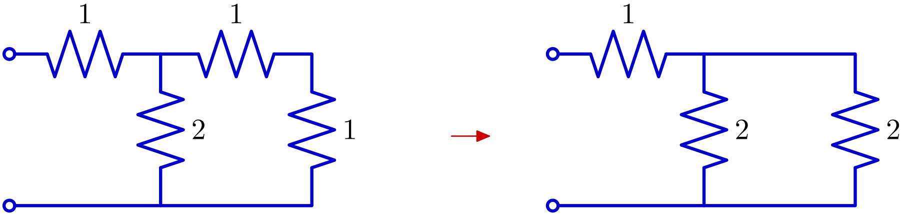
What is the equivalent resistance of the following one-port.



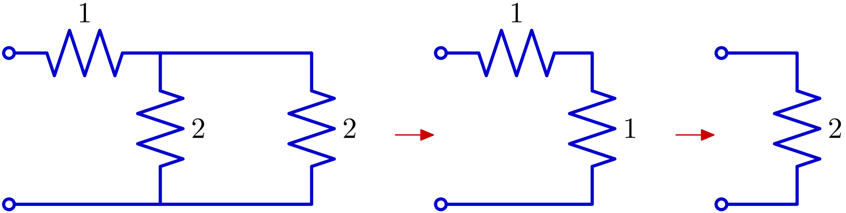
1. 0.5
2. 1
3. 2
4. 3
5. 5

Check Yourself

Combine two rightmost resistors (series):

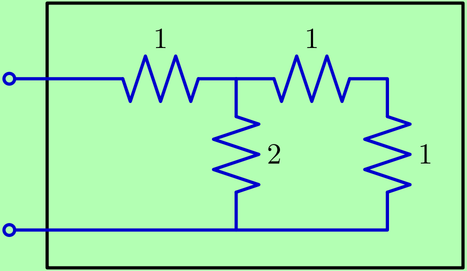


Combine rightmost parallel resistors, then the resulting series.



Check Yourself

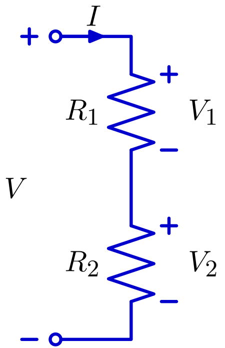
What is the equivalent resistance of the following one-port.



1. 0.5
2. 1
3. 2
4. 3
5. 5

Voltage Divider

Resistors in series act as voltage dividers.



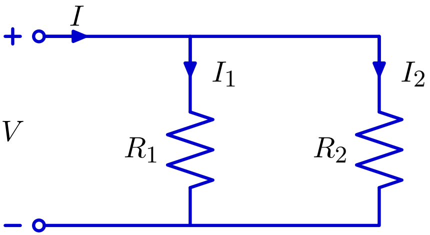
$$I = \frac{V}{R_1 + R_2}$$

$$V_1 = R_1 I = \frac{R_1}{R_1 + R_2} V$$

$$V_2 = R_2 I = \frac{R_2}{R_1 + R_2} V$$

Current Divider

Resistors in parallel act as current dividers.

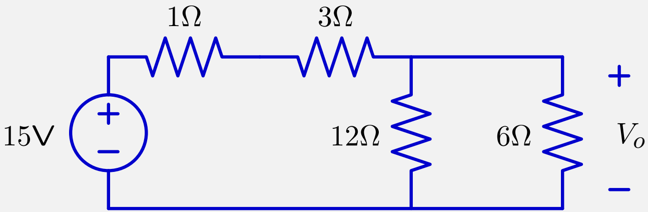


$$V = (R_1 || R_2)\, I$$

$$I_1 = \frac{V}{R_1} = \frac{R_1||R_2}{R_1}\, I = \frac{1}{R_1}\frac{R_1 R_2}{R_1 + R_2}\, I = \frac{R_2}{R_1 + R_2}\, I$$

$$I_2 = \frac{V}{R_2} = \frac{R_1||R_2}{R_2}\, I = \frac{1}{R_2}\frac{R_1 R_2}{R_1 + R_2}\, I = \frac{R_1}{R_1 + R_2}\, I$$

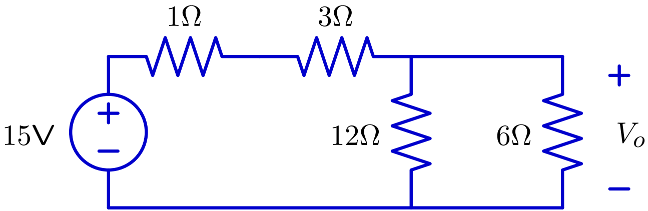
Check Yourself



Which of the following is true?

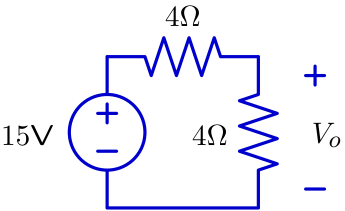
1. $V_o \leq 3$V
2. 3V$< V_o \leq 6$V
3. 6V$< V_o \leq 9$V
4. 9V$< V_o \leq 12$V
5. $V_o > 12$V

Check Yourself



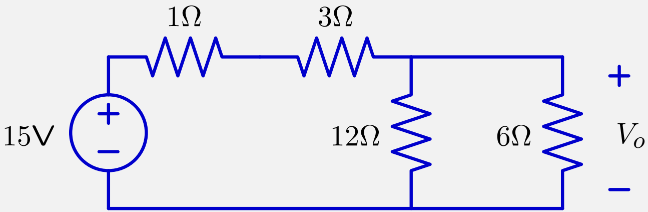
Add the top two resistances to get the series equivalent: 4Ω.

Then find the parallel equivalent: $\dfrac{12\Omega \times 6\Omega}{12\Omega + 6\Omega} = 4\Omega$.



Now apply the voltage divider relation: $V_o = \dfrac{4\Omega}{4\Omega + 4\Omega} \times 15\text{V} = 7.5\text{V}$.

Check Yourself



Which of the following is true? 3

1. $V_o \leq 3\text{V}$
2. $3\text{V} < V_o \leq 6\text{V}$
3. $6\text{V} < V_o \leq 9\text{V}$
4. $9\text{V} < V_o \leq 12\text{V}$
5. $V_o > 12\text{V}$

Summary

Circuits represent systems as connections of elements

- through which currents (through variables) flow and
- across which voltages (across variables) develop.

We have seen three (of many) methods for **analyzing** circuits.
Each one is based on a different set of variables:

- currents and voltages for each element
- node voltages
- loop currents

We can simplify analysis by recognizing common **patterns**:

- series and parallel combinations
- voltage and current dividers