Chapter 10: Sinusoids and Phasors

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4. Phasor Relationships for Circuit Elements
5. Impedance and Admittance
6. Kirchhoff's Laws in the Frequency Domain
7. Impedance Combinations
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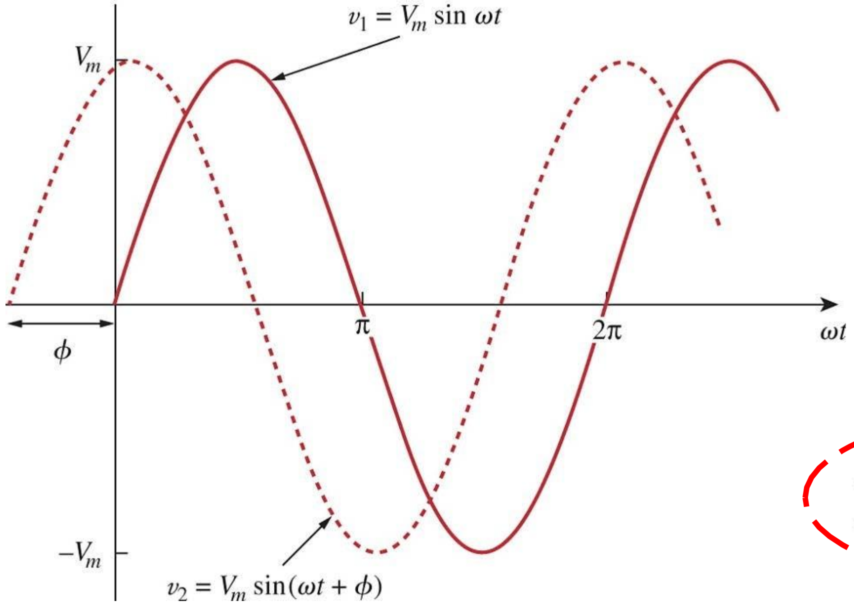
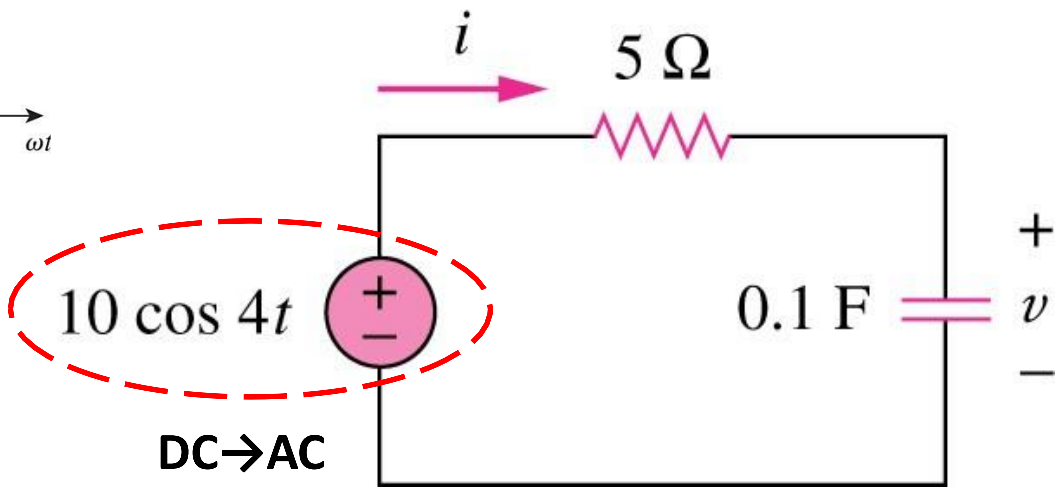
10.1 Motivation (1)

How to determine *v*(*t*) and *i*(*t*)?



$v_s(t) = 10$ V



DC→AC

How can we apply what we have learned before to determine $i(t)$ and $v(t)$?

10.1 Motivation (2)

- The output of the circuit will consist of two parts:
  - ✓a *transient part* that dies out as time increases;
  - ✓a *steady-state part* that persists.

Typically, the transient part dies out quickly, perhaps in a couple of milliseconds.

- AC circuits are the subject of this chapter. In particular,
  - ✓ It's useful to associate a complex number with a sinusoid. Doing so allows us to define phasors and impedances.
  - ✓Using phasors and impedances, we obtain a new representation of the linear circuit, called the "frequency-domain representation."
  - ✓We analyze ac circuits in the frequency domain to determine their steady-state response.

10.2 Sinusoids (1)

- A **sinusoid** is a signal that has the form of the sine or cosine function.
- A general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi)$$

where

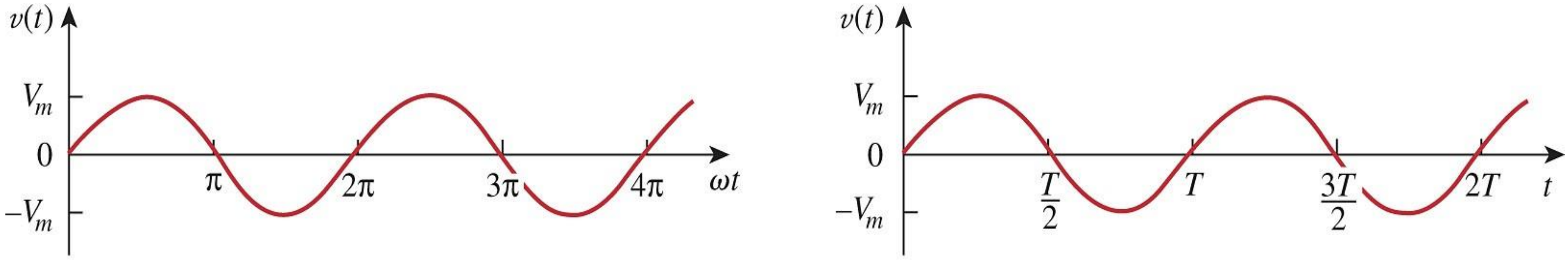
V_m = the **amplitude** of the sinusoid

ω = the **angular frequency** in radians/s

ωt = the **argument** of the sinusoid

ϕ = the phase

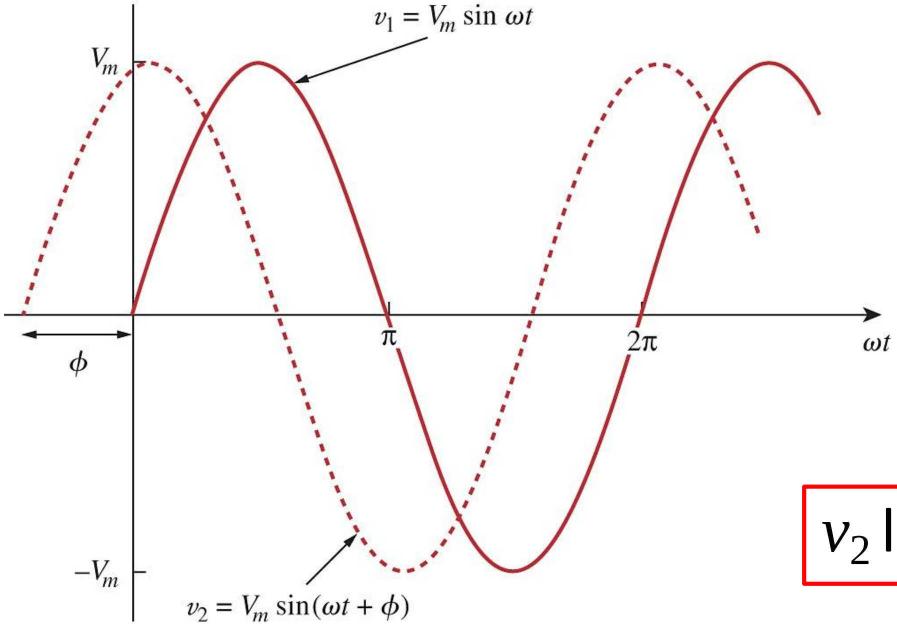
$$\text{period: } T = \frac{2\pi}{\omega}$$



10.2 Sinusoids (2)

A **periodic function** is one that satisfies $v(t) = v(t + nT)$, for all t and for all integers n.

$$v(t+T) = V_m \sin\omega(t+T) = V_m \sin\omega\left(t + \frac{2\pi}{\omega}\right)$$
$$= V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t)$$



$$f = \frac{1}{T} Hz \qquad \omega = 2\pi f$$

v_2 leads v_1 by ϕ or v_1 lags v_2 by ϕ

- Only two sinusoidal values with the **same frequency** can be compared by their amplitude and phase difference.
- If phase difference is zero, they are **in phase**; if phase difference is not zero, they are **out of phase**.

10.2 Sinusoids (3)

Example:

Given a sinusoid, $5\sin(4\pi t - 60°)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Solution:

Amplitude = 5, phase = –60°, angular frequency = 4 π rad/s, period = 0.5 s, frequency = 2 Hz.

Example:

Find the phase angle between $i_2 = 5\cos(377t - 40^o)$ and $i_1 = -4\sin(377t + 25^o)$, does i_1 lead or lag i_2?

Solution:

Since $\sin(\omega t + 90^o) = \cos \omega t$

i_1 leads i_2 by 155°.

10.3 Phasor (1)

- A **phasor** is a complex number that represents the amplitude and phase of a sinusoid.

$$i(t) = \mathrm{Re}\{I_{\mathrm{m}} e^{j\phi} e^{j\omega t}\} \Rightarrow \mathbf{I} = I_{\mathrm{m}} e^{j\phi} = I_{\mathrm{m}} \angle \phi$$

  where **I** is called a phasor.

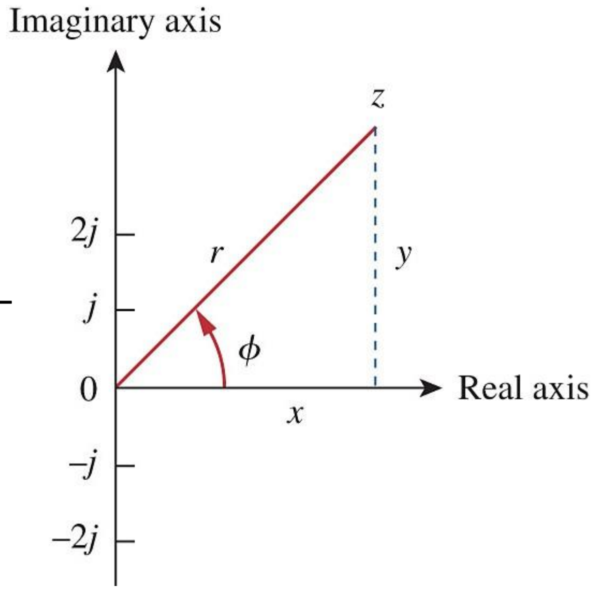
- Phasors may be used when the circuit is linear, the steady-state response is sought, and all independent sources are sinusoidal and have the same frequency.

- It can be represented in one of the following

  a. Rectangular $z = x + jy = r(\cos\phi + j\sin\phi)$

  b. Polar $z = r\angle\phi$

  c. Exponential $z = re^{j\phi}$ where $r = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}\dfrac{y}{x}$



10.3 Phasor (2)

Mathematic operation of complex number:

1. Addition $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
3. Multiplication $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$
4. Division $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \angle \phi_1 - \phi_2$
5. Reciprocal $\dfrac{1}{z} = \dfrac{1}{r} \angle -\phi$
6. Square root $\sqrt{z} = \sqrt{r} \angle \phi/2$
7. Complex conjugate $z^* = x - jy = r \angle -\phi = re^{-j\phi}$
8. Euler's identity $e^{\pm j\phi} = \cos\phi \pm j \sin\phi$

10.3 Phasor (3)

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

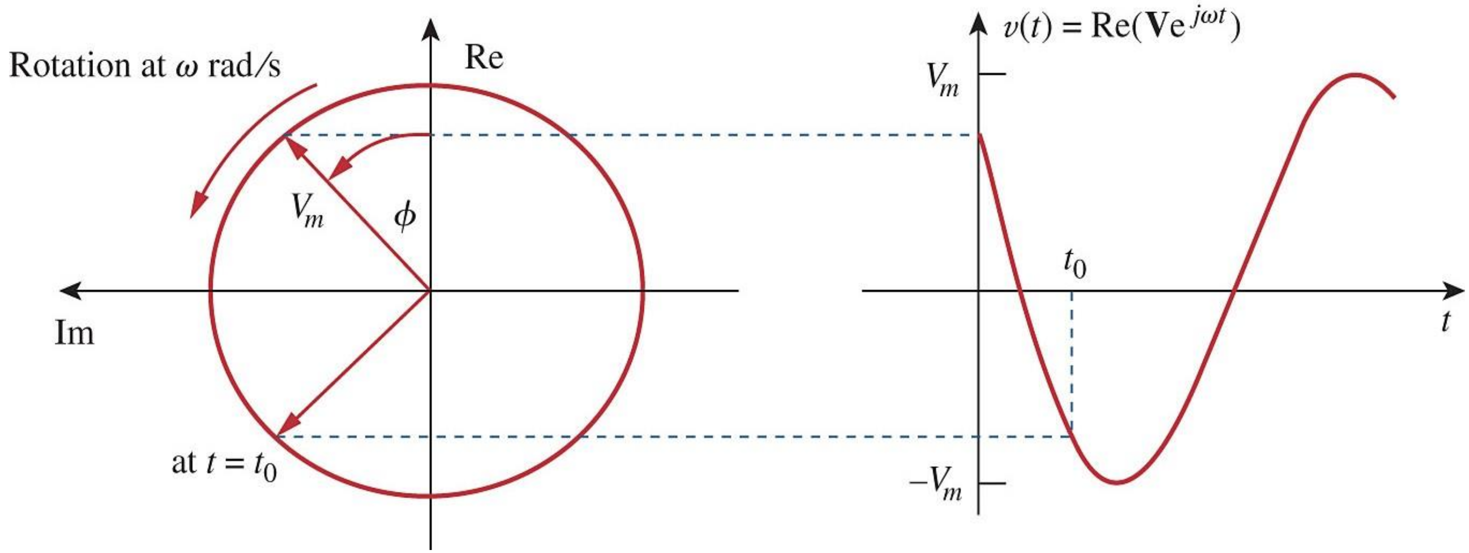
$$\cos\phi = \mathrm{Re}(e^{j\phi})$$
$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

$$v(t) = V_m\cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t+\phi)})$$

$$v(t) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t})$$

$$v(t) = \mathrm{Re}(\mathbf{V} e^{j\omega t})$$

$$\rightarrow \quad \mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi}$$



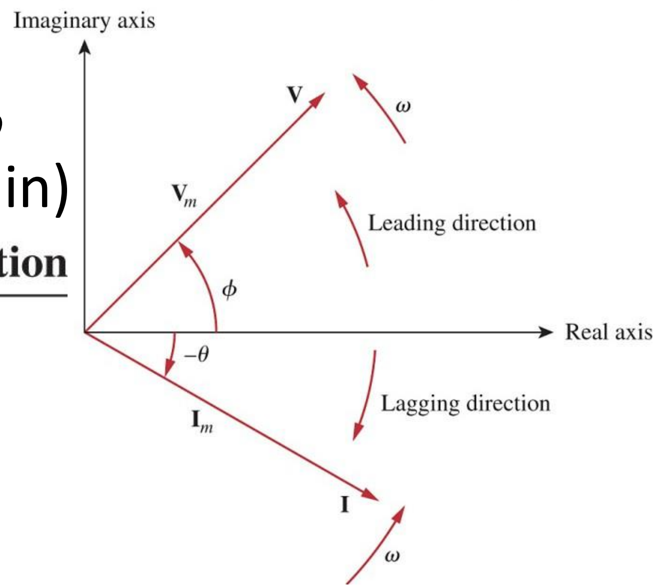
10.3 Phasor (4)

- Transform a sinusoid from the time domain to the phasor domain:

$$v(t) = V_m \cos(\omega t + \phi) \longleftrightarrow \mathbf{V} = V_m \angle \phi$$

(time domain) (phasor domain)

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$



- **Amplitude** and **phase difference** are two principal concerns in the study of voltage and current sinusoids.
- Phasor will be defined from the **cosine function** in all our proceeding study. If a voltage or current expression is in the form of a sine, it will be changed to a cosine by subtracting from the phase.

10.3 Phasor (5)

$v(t)$	$\mathbf{V} = V\angle\phi$	
$\dfrac{dv}{dt}$	$j\omega\mathbf{V}$	
$\int v dt$	$\dfrac{\mathbf{V}}{j\omega}$	**No initial cond. & set** $s = j\omega$

Laplace transform

$$L[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0^-)$$

$$\mathcal{L}\left[\int_0^t f(t)\, dt\right] = \frac{1}{s}F(s)$$

The differences between $v(t)$ and V:

- $v(t)$ is instantaneous or time-domain representation
  V is the frequency or phasor-domain representation.
- $v(t)$ is time dependent, **V** is not.
- $v(t)$ is always real with no complex term, **V** is generally complex.

Note: Phasor analysis applies only when frequency is constant; when it is applied to two or more sinusoid signals only if they have the **same frequency**.

10.3 Phasor (6)

Example: Use phasor approach, determine the current $i(t)$ in a circuit described by the integro-differential equation.

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75°)$$

Solution:

$$4\mathbf{I} + \frac{8\mathbf{I}}{j\omega} - 3j\omega\mathbf{I} = 50\angle 75°$$

$a_1 = 50;\ a_2 = 4.642$

$\phi_1 = 75°;\ \phi_2 = 143.2°$

But $\omega = 2$, so

$$\mathbf{I}(4 - j4 - j6) = 50\angle 75°$$

$$\mathbf{I} = \frac{50\angle 75°}{4 - j10} = \frac{50\angle 75°}{10.77\angle -68.2°} = 4.642\angle 143.2°\text{ A}$$

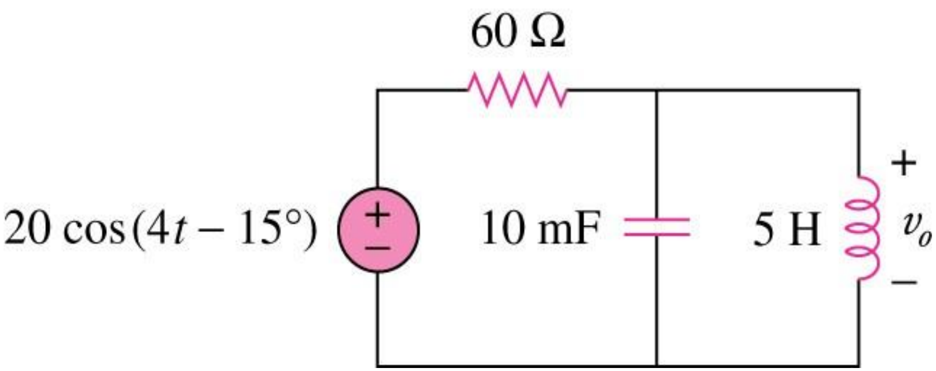
Converting this to the time domain,

$$i(t) = 4.642\cos(2t + 143.2°)\text{ A}$$

Keep in mind that this is only the steady-state solution, and it does not require knowing the initial values.

10.3 Phasor (7)

- We can derive the differential equations for the following circuit in order to solve for $v_o(t)$ in phase domain $\mathbf{V}_o$.



$$\frac{d^2 v_o}{dt^2} + \frac{5}{3}\frac{dv_0}{dt} + 20v_0 = -\frac{400}{3}\sin(4t - 15^o)$$

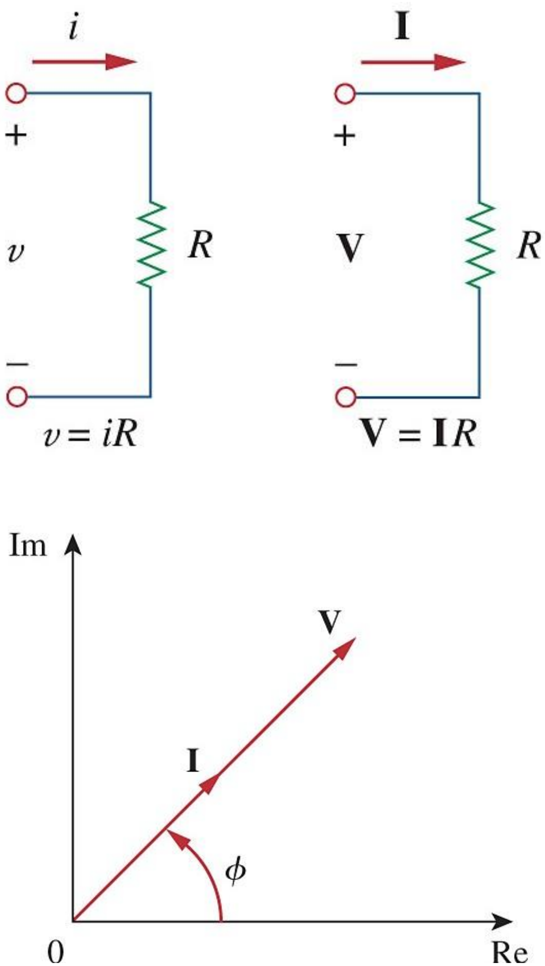
$$\frac{d^2 f}{dt^2} \rightarrow s^2F(s) - sf(0^-) - f'(0^-)$$

However, the derivation may sometimes be very tedious.

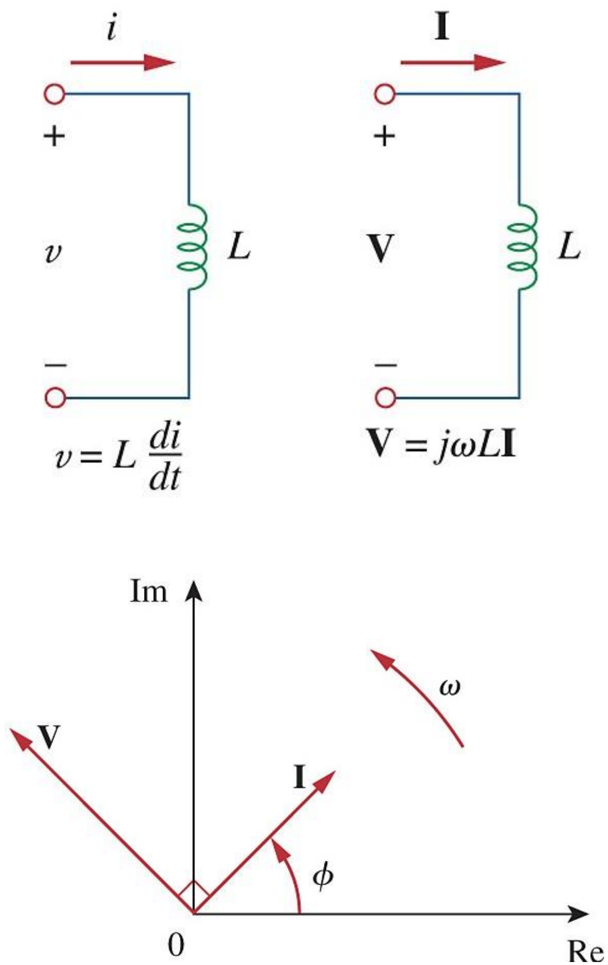
- Instead of first deriving the differential equation and then transforming it into phasor to solve for $\mathbf{V}_{o,}$ we can transform all the *RLC* components into phasor first, then apply the *KCL* laws and other theorems to set up a phasor equation involving $\mathbf{V}_o$ directly.

10.4 Phasor Relationships for Circuit Elements (1)

Resistor:

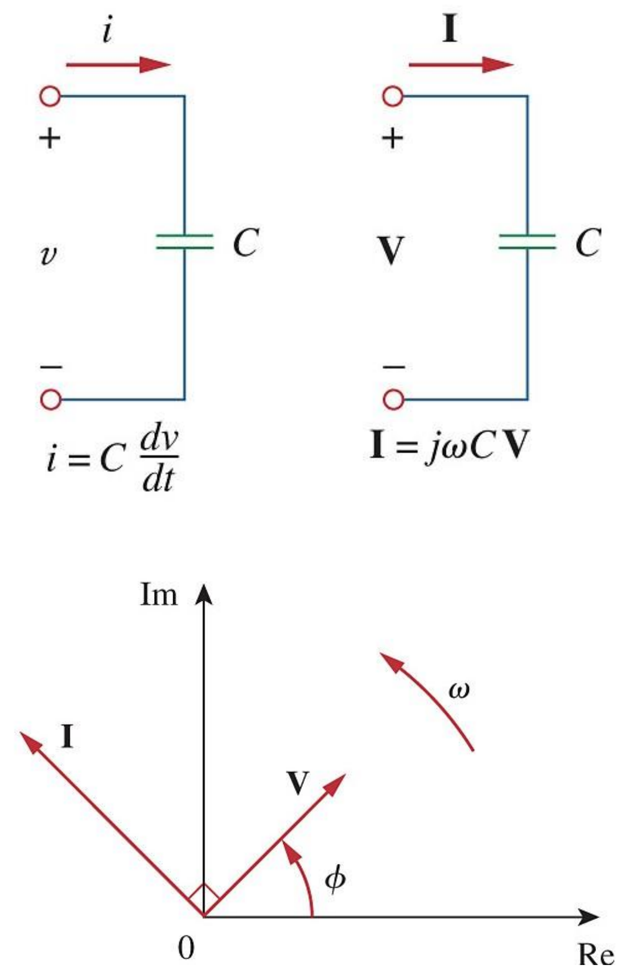


Inductor:



V leads **I** by 90°

Capacitor:



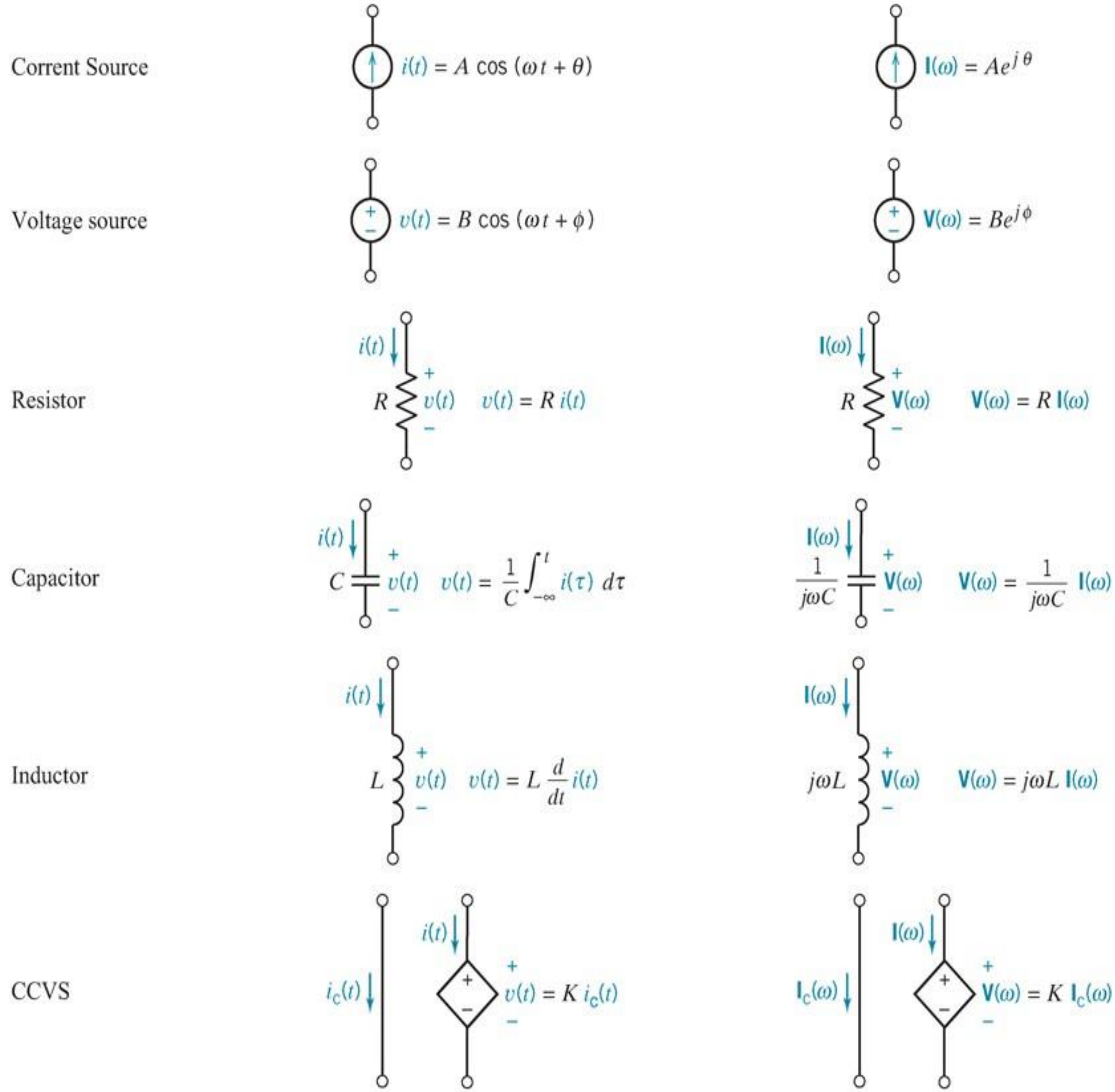
I leads **V** by 90°

10.4 Phasor Relationships for Circuit Elements (2)

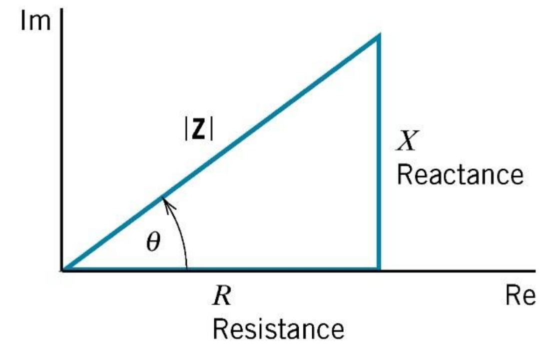
Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Example: If voltage $v(t) = 6\cos(100t - 30^o)$ is applied to a 50 µF capacitor, calculate the current, $i(t)$, through the capacitor.

Answer: $i(\mathrm{t}) = 30\cos(100t + 60^o)$ mA


Corrent Source
$i(t) = A\cos(\omega t + \theta)$
$\mathbf{I}(\omega) = Ae^{j\theta}$
Voltage source
$v(t) = B\cos(\omega t + \phi)$
$\mathbf{V}(\omega) = Be^{j\phi}$
Resistor
$v(t) = R\,i(t)$
$\mathbf{V}(\omega) = R\,\mathbf{I}(\omega)$
Capacitor
$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau$
$\mathbf{V}(\omega) = \frac{1}{j\omega C}\mathbf{I}(\omega)$
Inductor
$v(t) = L\frac{d}{dt}i(t)$
$\mathbf{V}(\omega) = j\omega L\,\mathbf{I}(\omega)$
CCVS
$v(t) = K\,i_c(t)$
$\mathbf{V}(\omega) = K\,\mathbf{I}_c(\omega)$

10.5 Impedance and Admittance (1)

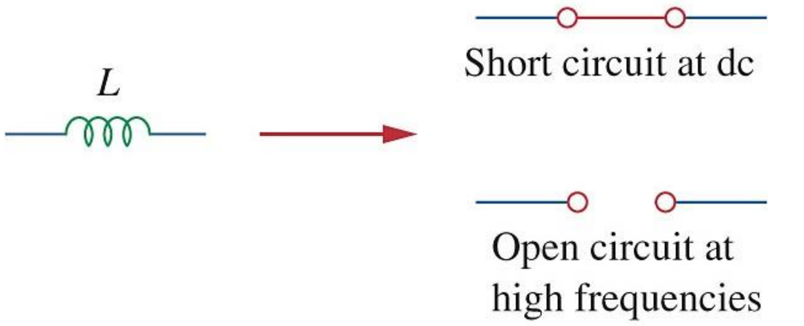


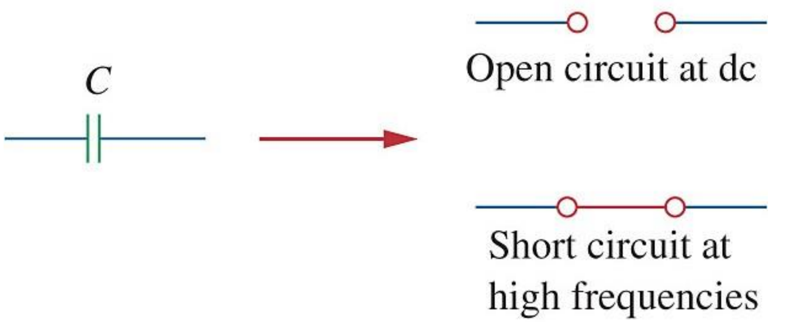
- The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms Ω.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R + jX$$

where $R = \text{Re}(\mathbf{Z})$ is the resistance and $X = \text{Im}(\mathbf{Z})$ is the reactance. Positive X is for L (or lagging) and negative X is for C (or leading).

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$





- The admittance **Y** is the reciprocal of impedance,

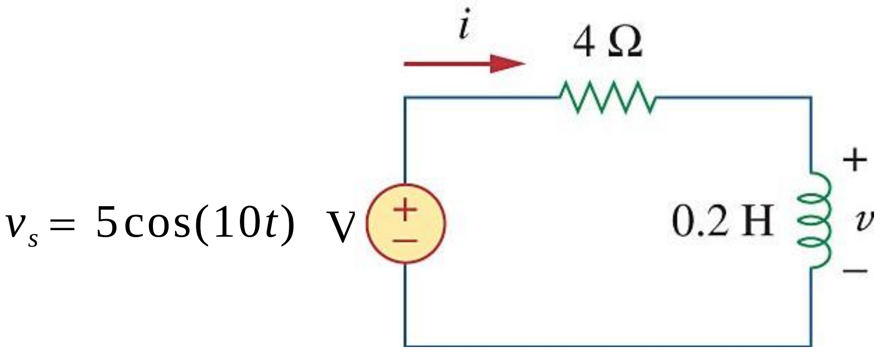
$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} = G + jB$$

Unit: siemens (S)

10.5 Impedance and Admittance (2)

- After we know how to convert *RLC* components from time to phasor domain, we can transform a time domain circuit into a phasor/frequency domain circuit.
- Hence, we can apply the *KCL* laws and other theorems to directly set up phasor equations involving our target variables for solving.

Example: Determine $v(t)$ and $i(t)$.



Answers: $i(t) = 1.118\cos(10t - 26.56^o)$ A; $v(t) = 2.236\cos(10t + 63.43^o)$ V

10.6 Kirchhoff's Laws in the Freq. Domain (1)

- Both *KVL* and *KCL* are hold in the phasor domain or more commonly called frequency domain.
- Moreover, the variables to be handled are phasors, which are complex numbers.
- All the mathematical operations involved are now in complex domain.

$$v_1 + v_2 + \cdots + v_n = 0$$

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0 \longleftarrow \text{same frequency}$$

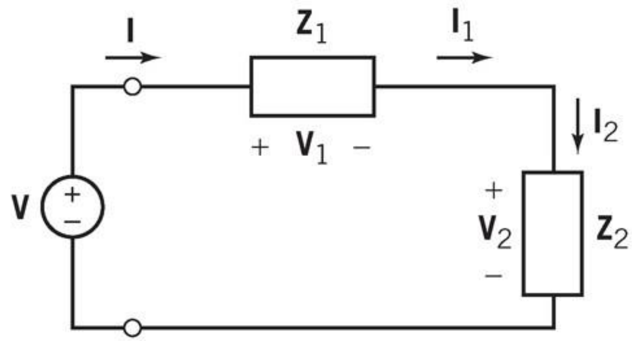
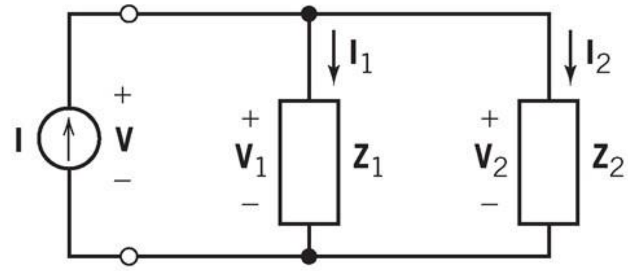
$$\text{Re}[(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2} + \cdots + V_{mn}e^{j\theta_n})e^{j\omega t}] = 0 \longleftarrow \text{complex numbers}$$

$$\text{Re}[(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n)e^{j\omega t}] = 0$$

$$\boxed{\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0}$$

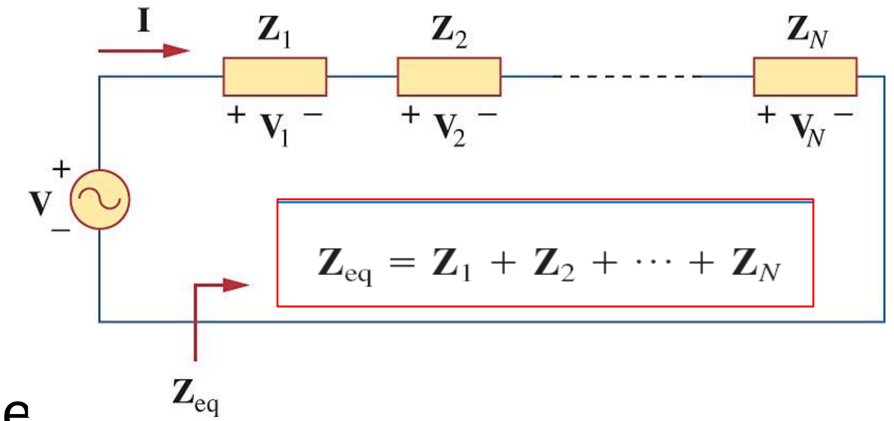
10.6 Kirchhoff's Laws in the Freq. Domain (2)

Table **Voltage and Current Division in the Frequency Domain**

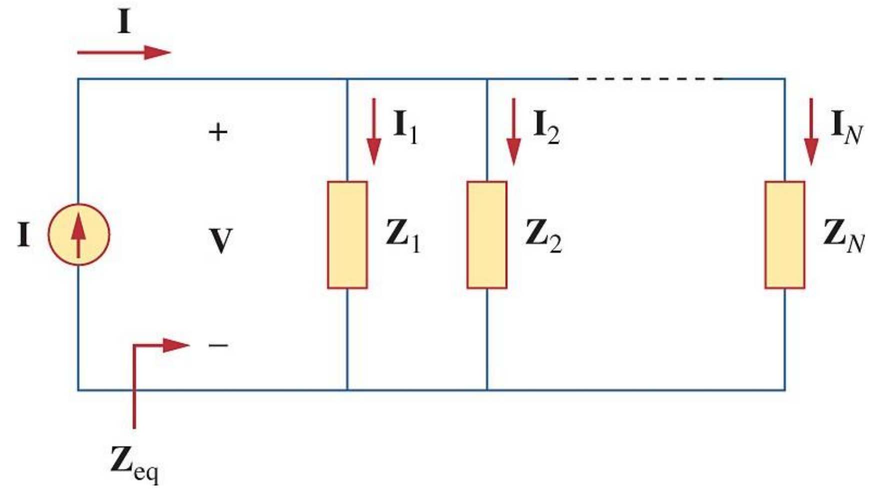
	CIRCUIT	EQUATIONS
Voltage division		$\mathbf{I}_1 = \mathbf{I}_2 = \mathbf{I}$ $\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$ $\mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$
Current division		$\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}$ $\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$ $\mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$

10.7 Impedance Combinations (1)

- The following principles used for DC circuit analysis all apply to AC circuit.
- For example:
  a. voltage division
  b. current division
  c. circuit reduction
  d. impedance equivalence



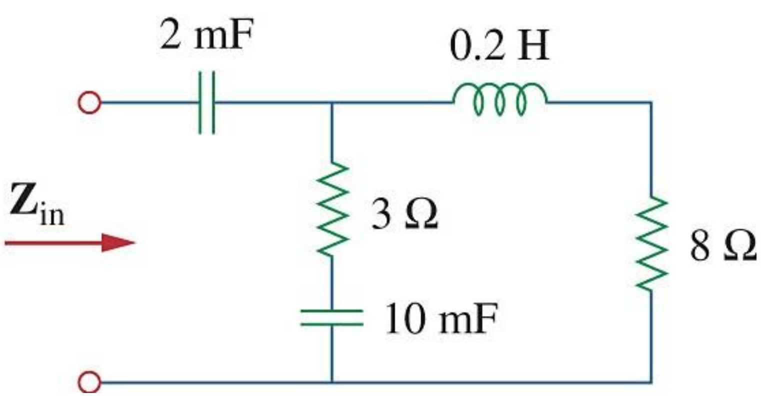
$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$



$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

10.7 Impedance Combinations (2)

Example: Find the input impedance at ω = 50 rad/s.



$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

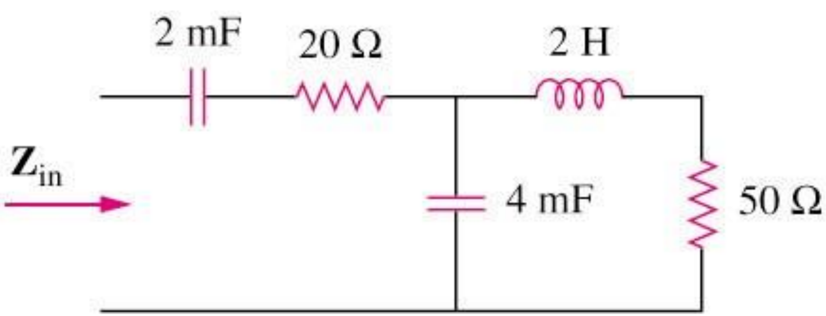
$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

Example: Determine the input impedance at ω = 10 rad/s.

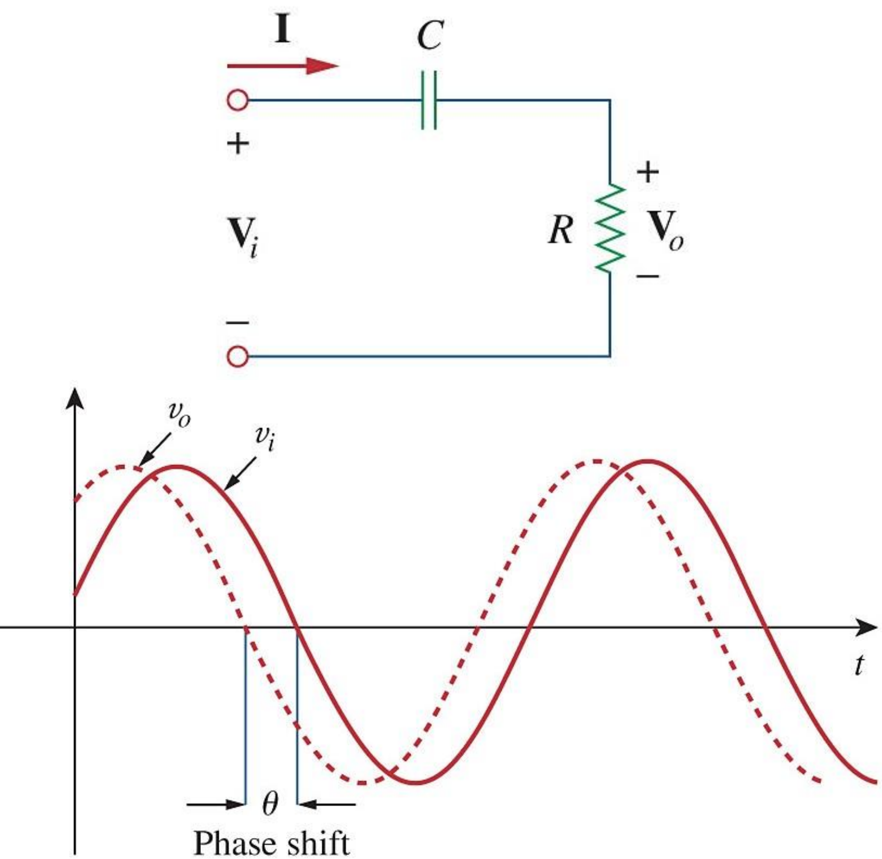


Answer: $\mathbf{Z}_{in}$ = $32.38 - j73.76$

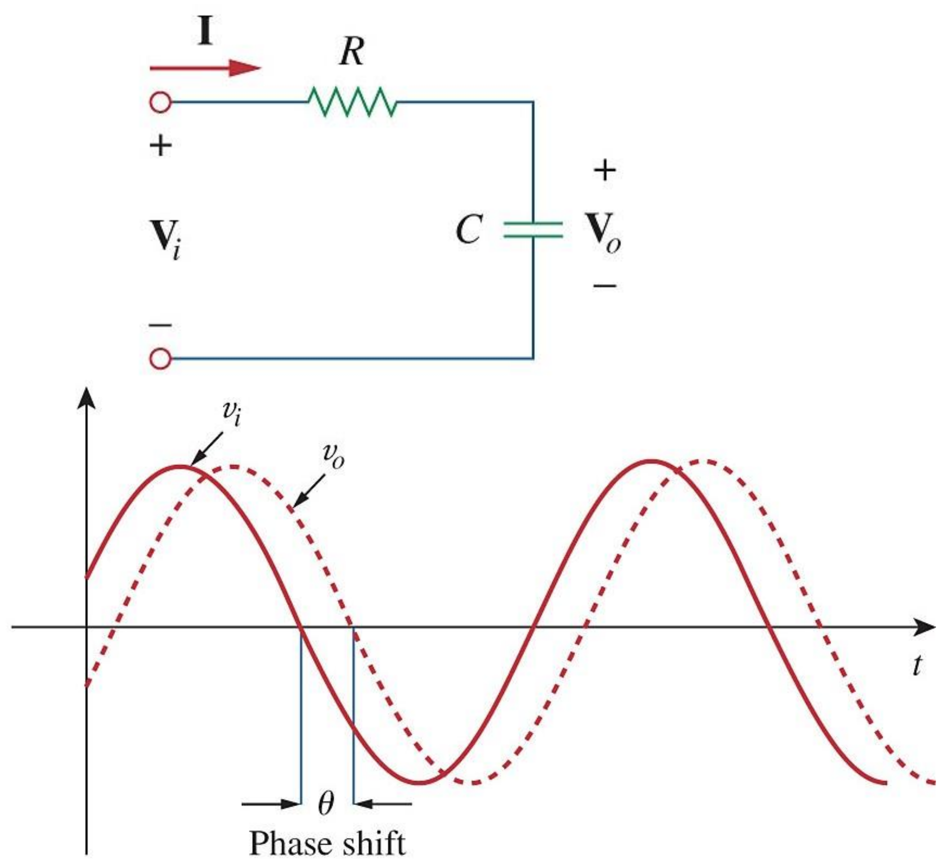
10.8 Application: Phase-Shifters

- Series RC shift circuits:

  - Leading output



  - Lagging output



$$\theta = \tan^{-1} \frac{X_C}{R}$$

10.9 Summary (1)

- With the pervasive use of ac electric power in the home and industry, it is important for engineers to analyze circuits with sinusoidal independent sources.
- The steady-state response of a linear circuit to a sinusoidal input is itself a sinusoid having the same frequency as the input signal.
- Circuits that contain inductors and capacitors are represented by differential equations. When the input to the circuit is sinusoidal, the phasors and impedances can be used to represent the circuit in the frequency domain. In the frequency domain, the circuit is represented by algebraic equations.
- The steady-state response of a linear circuit with a sinusoidal input is obtained as follows:
  1. Transform the circuit into the frequency domain, using phasors and impedances.

10.9 Summary (2)

2. Represent the frequency-domain circuit by algebraic equation, for example, mesh or node equations.
3. Solve the algebraic equations to obtain the response of the circuit.
4. Transform the response into the time domain, using phasors.

- A circuit contains several sinusoidal sources, two cases:
  - ✓ When all of the sinusoidal sources have the same frequency, the response will be a sinusoid with that frequency, and the problem can be solved in the same way that it would be if there was only one source.
  - ✓ When the sinusoidal sources have different frequencies, superposition is used to break the time-domain circuit up into several circuits, each with sinusoidal inputs all at the same frequency. Each of the separate circuits is analyzed separately and the responses are summed in the time domain.