Portland State University

ECE241

Introduction to Electrical Engineering

Prefixes

Prefixes modify some basic unit. Engineering notation refers to those prefixes where the exponent is a multiple of three. For example, the most commonly used prefixes for resistance (Ohm), inductance (Henry), and capacitance (Farad) are:

Prefix	Symbol	Power of Ten	Ohm	Henry	Farad
pico	p	10^{-12}			pF
nano	n	10^{-9}			nF
micro	μ	10^{-6}		μH	μF
milli	m	10^{-3}		mH	
—	—	—	Ω	H	
kilo	k	10^{3}	kΩ		
mega	M	10^{6}	MΩ		

Voltage and Current

Voltage is created by the separation of charge, and current is created by the movement of charge. By definition,

$$\boxed{v = \frac{dw}{dq}} \qquad \boxed{i = \frac{dq}{dt}}$$

where

v = voltage in volts i = current in amperes

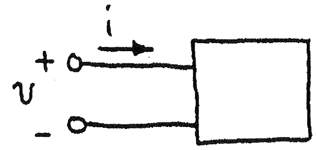
w = energy in joules q = charge in coulombs

q = charge in coulombs t = time in seconds

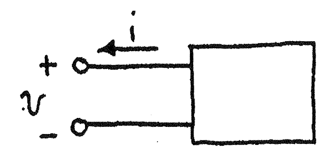
These definitions of voltage and current describe magnitude but not polarity. Voltage requires a reference polarity (plus and minus) and current requires a reference direction for positive charge movement.

Basic Circuit Elements

Basic circuit elements are described in terms of the circuit variables voltage and/or current. The assignment of the reference polarity for voltage and the reference direction for current is completely arbitrary. However, all equations written must agree with the chosen references. In addition, the passive sign convention must be observed:



positive sign
with
voltage/current expression



negative sign
with
voltage/current expression

Power and Energy

Power is defined as the time rate of expending or absorbing energy. By definition,

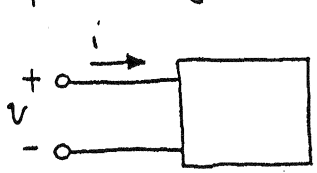
$$\boxed{p = \frac{dw}{dt}} \qquad \boxed{p = vi}$$

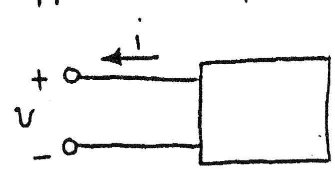
where

p = power in watts
w = energy in joules
t = time in seconds

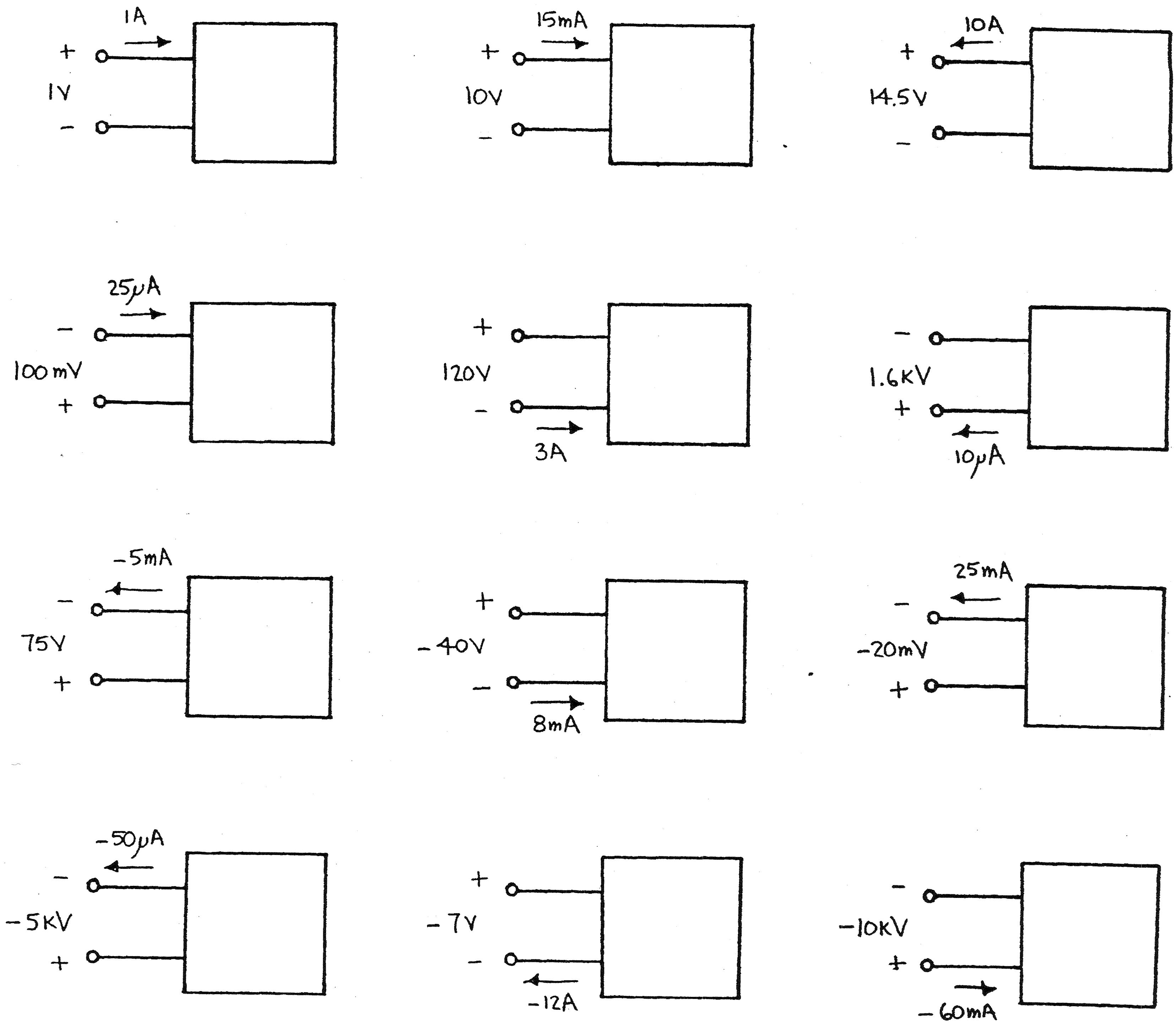
p = power in watts
v = voltage in volts
i = current in amperes

The passive sign convention applied to the power equation is as follows.



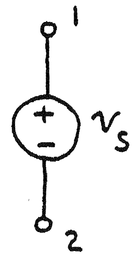


$$\boxed{p = +vi} \qquad \boxed{p = -vi}$$

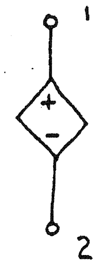

1A
+
1V
−
15mA
+
10V
−
10A
+
14.5V
−
25μA
−
100mV
+
+
120V
−
3A
−
1.6KV
+
10μA
−5mA
−
75V
+
+
−40V
−
8mA
25mA
−
−20mV
+
−50μA
−
−5KV
+
+
−7V
−
−12A
−
−10KV
+
−60mA

Voltage and Current Sources

An ideal voltage source is a circuit element that produces v_s volts regardless of the current in the device. The symbols are



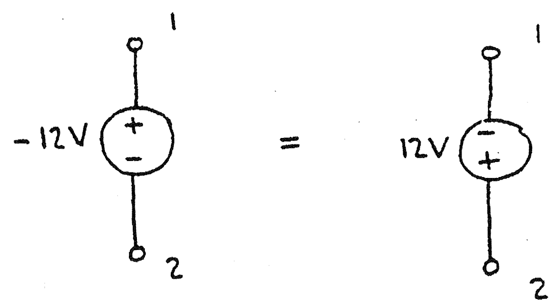
independent voltage source



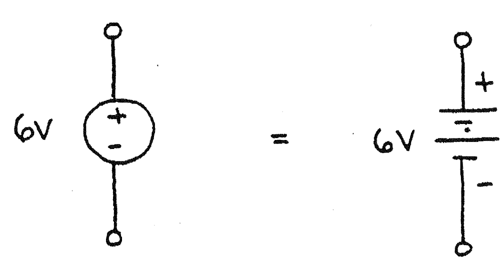
$$v_s = \begin{cases} \mu v_x \\ \rho i_x \end{cases}$$

dependent voltage source

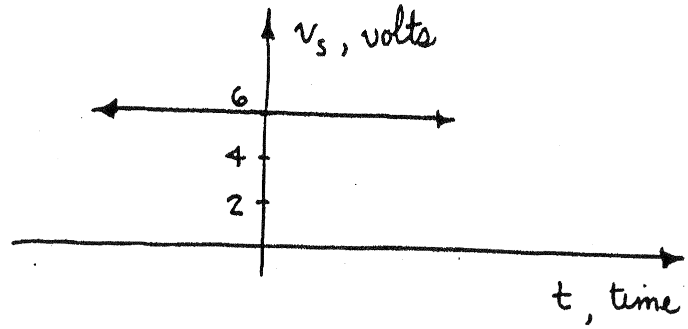
which denotes that terminal 1 is v_s volts higher than that of terminal 2. However, the quantity v_s can be either a positive or negative number. If v_s is negative, the polarity of the voltage can be reversed to achieve an equivalent voltage source where v_s is positive.



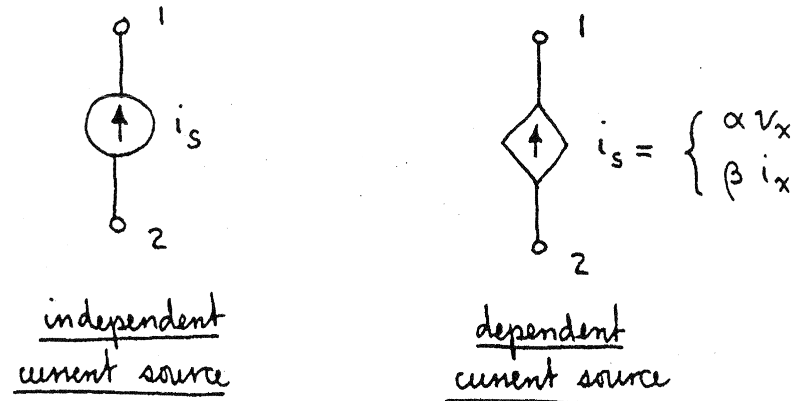
Another symbol that is sometimes used when v_s is a constant is that of the ideal battery.



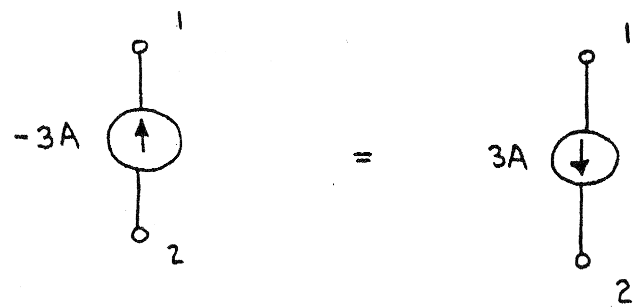
The graph of voltage versus time for this device is



An <u>ideal current source</u> is a circuit element that produces i_s amperes regardless of the voltage across the device. The symbols are

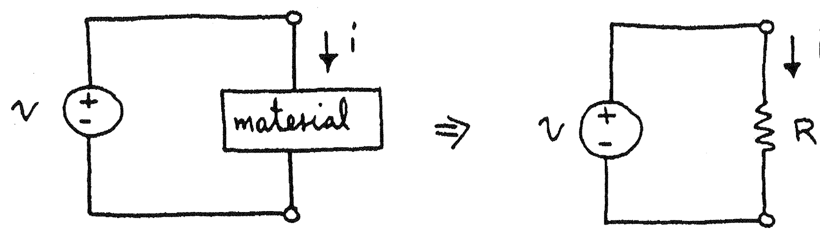


which denotes that i_s amperes flows from terminal 2 to terminal 1. If i_s is negative, the direction of the current can be reversed to achieve an equivalent current source where i_s is positive.

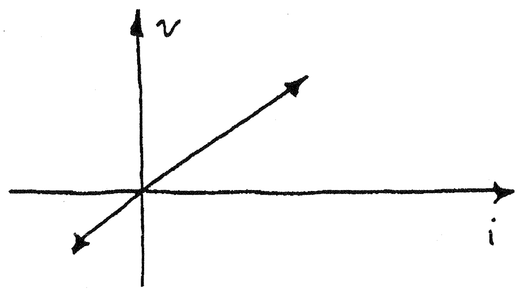


Electrical Resistance and Ohm's Law

Consider the following circuit.



The graph of v versus i is

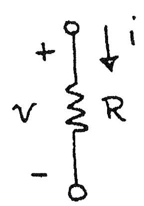


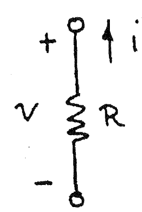
The proportionality constant for this straight line is called <u>resistance</u> R. The unit of resistance is the <u>ohm</u>. Therefore,

$$\boxed{v = iR}$$

and this equation is called <u>Ohm's law</u>.

The passive sign convention applied to Ohm's law is as follows.

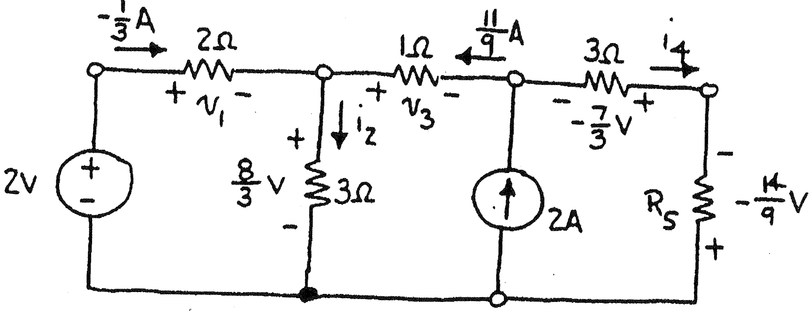




$$\boxed{v = +iR} \qquad \boxed{v = -iR}$$

Example:

Use Ohm's law to find the missing quantities.



(a) $v_1 = +\,i_1 R_1 = +\left(-\frac{1}{3}\right)(2) = \underline{\underline{-\frac{2}{3}\text{V}}}$ ←

(b) $i_2 = +\dfrac{v_2}{R_2} = +\dfrac{\frac{8}{3}}{3} = \underline{\underline{\frac{8}{9}\text{A}}}$ ←

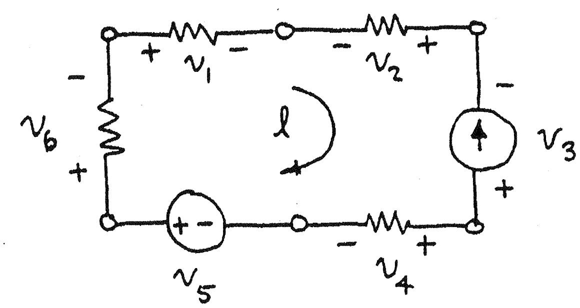
(c) $v_3 = -\,i_3 R_3 = -\left(\frac{11}{9}\right)(1) = \underline{\underline{-\frac{11}{9}\text{V}}}$ ←

(d) $i_4 = -\dfrac{v_4}{R_4} = -\dfrac{-\frac{7}{3}}{3} = \underline{\underline{\frac{7}{9}\text{A}}}$ ←

(e) $R_5 = -\dfrac{v_5}{i_5} = -\dfrac{-\frac{14}{9}}{\frac{7}{9}} = \underline{\underline{2\Omega}}$ ←

Kirchhoff's Voltage Law

KVL: The algebraic sum of all voltages around a loop is zero.

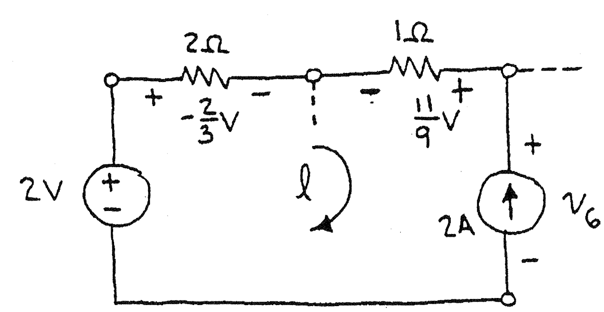


$$\sum_{l} v = v_1 - v_2 - v_3 + v_4 - v_5 + v_6 = 0$$

$$v_1 + v_4 + v_6 = v_2 + v_3 + v_5$$

Example:

Consider a loop from the previous example.



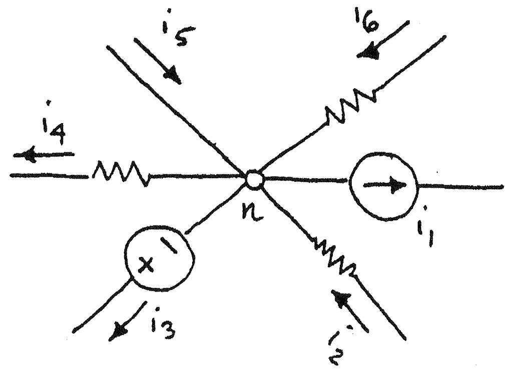
$$\sum_{l} v = +\left(-\frac{2}{3}\right) - \frac{11}{9} + v_6 - 2 = 0$$

$$v_6 = \frac{6}{9} + \frac{11}{9} + \frac{18}{9}$$

$$= \underline{\underline{\frac{35}{9}\text{V}}} \leftarrow$$

Kirchhoff's Current Law

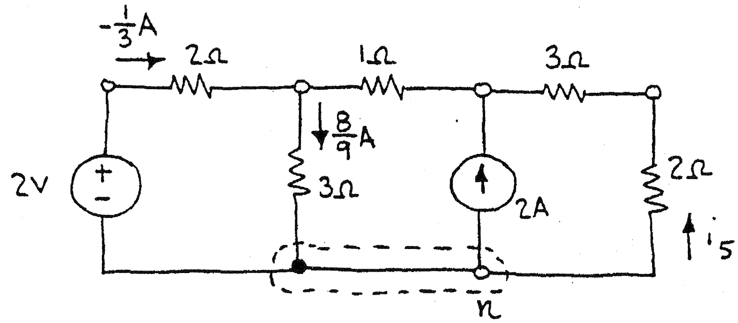
KCL: The algebraic sum of all currents around a node is zero.



$$\sum_n i = i_1 - i_2 + i_3 + i_4 - i_5 - i_6 = 0$$

$$i_1 + i_3 + i_4 = i_2 + i_5 + i_6$$

Example:



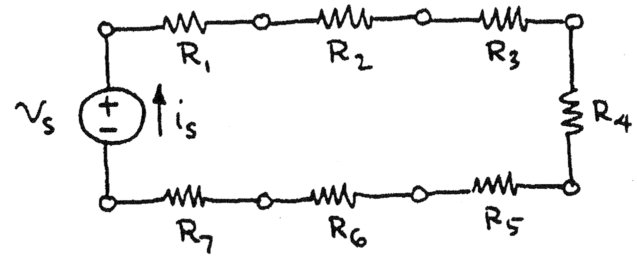
$$\sum_n i = +\left(-\tfrac{1}{3}\right) - \tfrac{8}{9} + 2 + i_5 = 0$$

$$i_5 = \frac{3}{9} + \frac{8}{9} - \frac{18}{9}$$

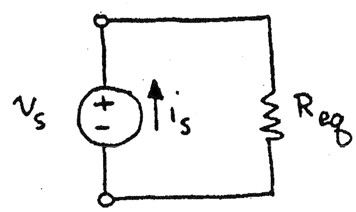
$$= -\frac{7}{9}\text{ A}$$

Resistors in Series

Circuit elements connected in series carry the same current. When the circuit elements are resistors, the circuit becomes



This circuit can be simplified to

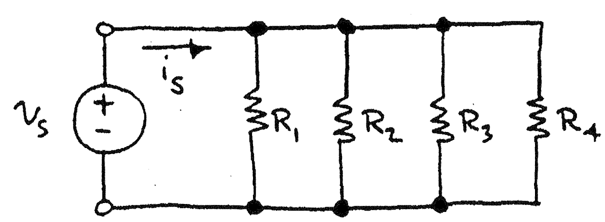


where

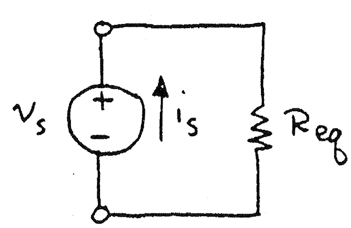
$$\boxed{R_{eq} = \sum_{i=1}^{K} R_i = R_1 + R_2 + \cdots + R_K}$$

Resistors in Parallel

Circuit elements connected in parallel have the same voltage. For resistors,



This circuit can also be simplified to



where

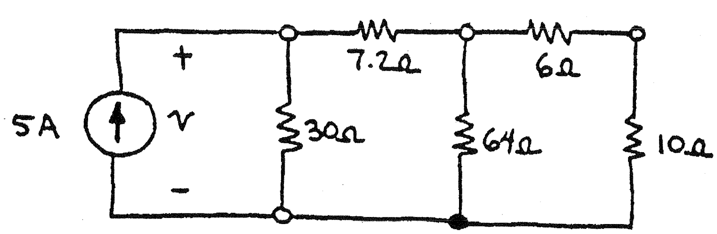
$$\boxed{\frac{1}{R_{eq}} = \sum_{i=1}^{K} \frac{1}{R_i} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_K}}$$

Also,

$$\boxed{G_{eq} = \sum_{i=1}^{K} G_i = G_1 + G_2 + \cdots + G_K}$$

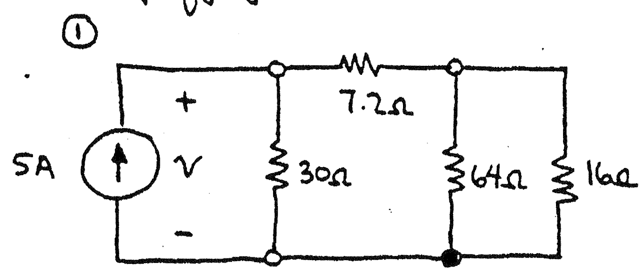
Example:

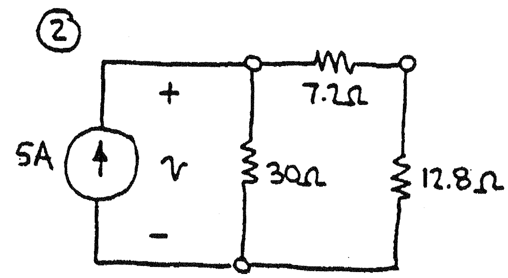
Consider the following circuit.

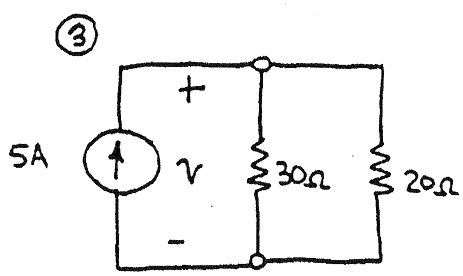


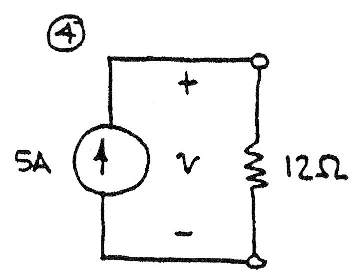
Find (a) the voltage v, (b) the power delivered by the current source, and (c) the power dissipated in the 10Ω resistor.

(a) Simplifying









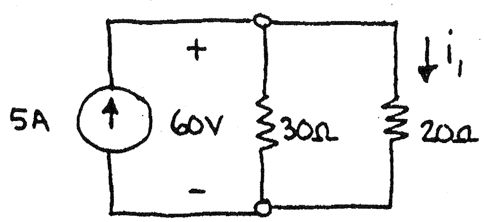
Using ohm's law,

$$v = i R_{eq} = (5)(12) = \underline{\underline{60V}}$$

(b)

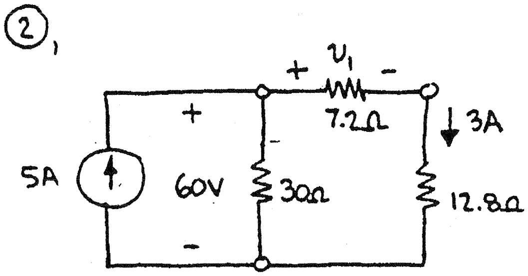
$$p = -vi = -(60)(5) = \underline{\underline{-300W}}$$

(c) Using ohm's law on circuit (3)



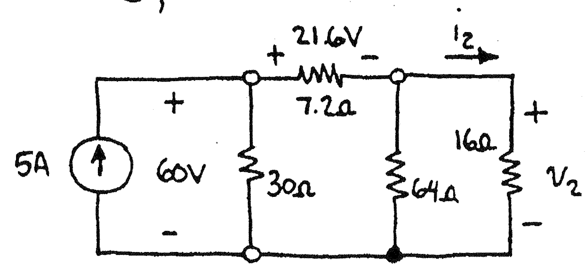
$$i_1 = \frac{V}{20} = \frac{60}{20} = \underline{3A}$$

And on (2),



$$v_1 = i_1(7.2) = (3)(7.2) = \underline{21.6V}$$

Using KVL on (1),



$$21.6 + v_2 - 60 = 0$$

$$v_2 = \underline{38.4V}$$

Also,

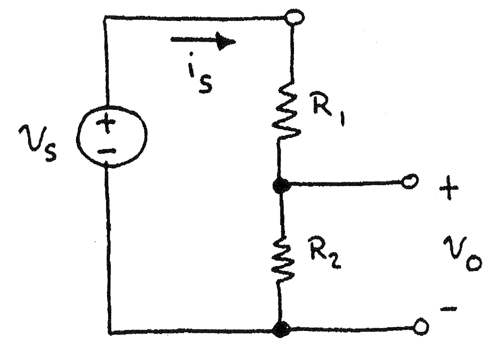
$$i_2 = \frac{v_2}{16} = \frac{38.4}{16} = \underline{2.4A}$$

Therefore,

$$P = (i_2)^2 10 = (2.4)^2(10) = \underline{\underline{57.6W}}$$

The Voltage Divider Circuit

The voltage divider circuit produces a fractional part of an applied voltage.



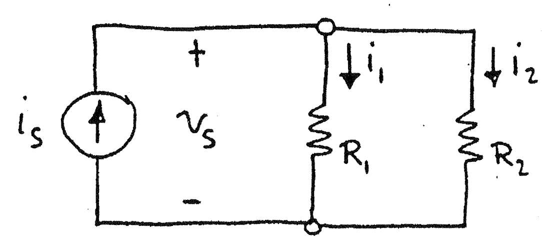
The voltage divider theorem is

$$\boxed{v_o = \frac{R_2}{R_1+R_2}\, v_s}$$

which permits the direct determination of v_o without determining i_s.

The Current Divider Circuit

The current divider circuit produces a fractional part of an incoming current.



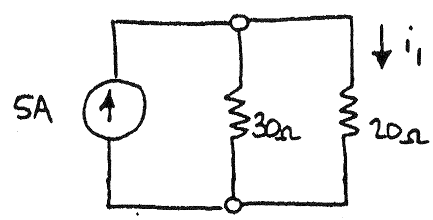
the current divider theorem is

$$\boxed{i_2 = \frac{R_1}{R_1+R_2}\, i_s}$$

which permits the direct determination of i_2 (or i_1) without determining v_s.

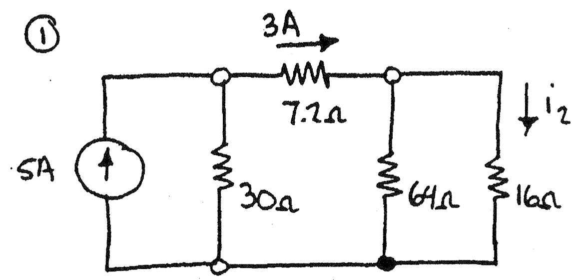
Using the previous example,

③



$$i_1 = \frac{30}{20+30}(5) = \underline{3A}$$

①



$$i_2 = \frac{64}{16+64}(3) = \underline{2.4A}$$

And,

$$p = (i_2)^2\, 10 = (2.4)^2(10) = \underline{\underline{57.6\,W}} \leftarrow$$

Network Terminology

Planar Circuit: A circuit that can be drawn on a plane with no crossing branches

Node: Point or portion of a circuit where 2 or more elements are joined

Essential Node: Point or portion of a circuit where 3 or more elements are joined

Branch: Path that connects 2 nodes

Essential Branch: Path that connects 2 essential nodes w/o passing through an essential node

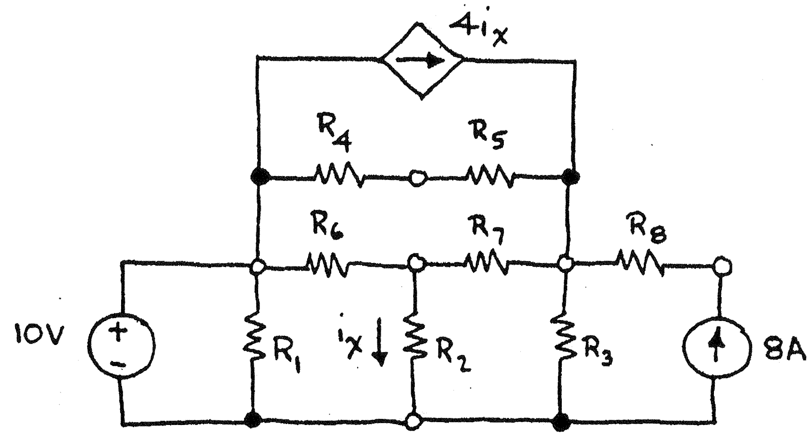
Loop: Path with last node same as starting node that does not cross itself

Mesh: Loop that does not enclose any other loops

Note: this isn't in the text.

Network Terminology

Consider the following planar circuit.

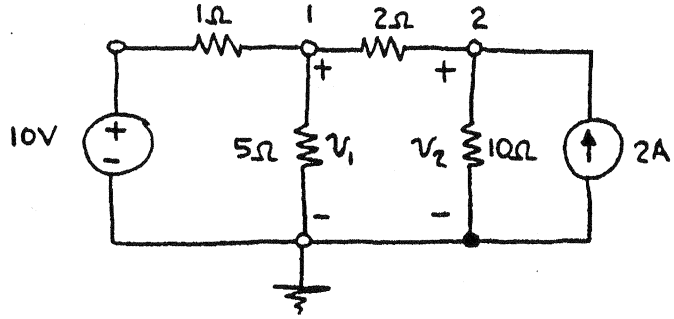


The following information is obtained :

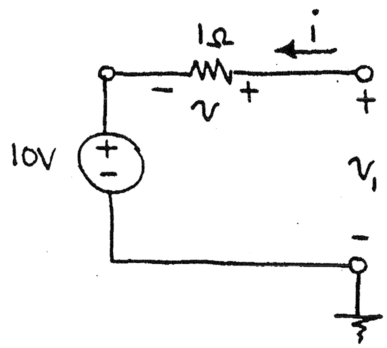
1) nodes = 6
2) essential nodes = 4
3) branches = 11
4) essential branches = 9
5) essential branches with unknown currents = 7
6) meshes = 6

Nodal Analysis

Consider the following circuit with a reference node.



Branch currents are now expressed in terms of node voltages and branch conductances (or resistances). For example,



using KVL,

$$v_1 - 10 - v = 0$$

$$v = v_1 - 10$$

therefore,

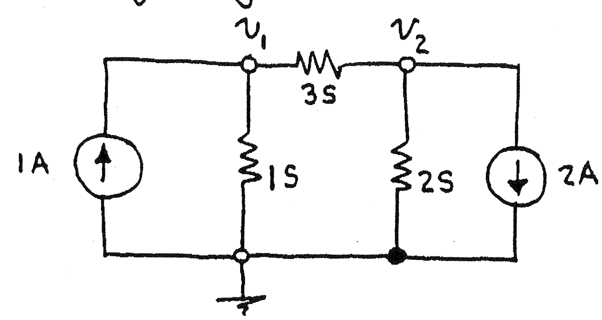
$$i = \frac{v}{R} = \frac{v_1 - 10}{1}$$

the nodal equations for the above circuit are

$$\begin{cases} \dfrac{v_1 - 10}{1} + \dfrac{v_1}{5} + \dfrac{v_1 - v_2}{2} = 0 \\[2ex] \dfrac{v_2 - v_1}{2} + \dfrac{v_2}{10} - 2 = 0 \end{cases}$$

Example:

Consider the following circuit.



Using KCL at node 1,

$$-1 + 1v_1 + 3(v_1 - v_2) = 0$$

$$4v_1 - 3v_2 = 1$$

At node 2,

$$3(v_2 - v_1) + 2v_2 + 2 = 0$$

$$-3v_1 + 5v_2 = -2$$

In matrix terms,

$$\begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$[G][v] = [i]$$

The solution using Cramer's rule is

$$v_1 = \frac{\begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ -3 & 5 \end{vmatrix}} = \frac{5-6}{20-9} = \underline{\underline{-\frac{1}{11}\,V}} \leftarrow$$

$$v_2 = \frac{\begin{vmatrix} 4 & 1 \\ -3 & -2 \end{vmatrix}}{11} = \frac{-8+3}{11} = \underline{\underline{-\frac{5}{11}\,V}} \leftarrow$$

If the inverse of matrix G is known,

$$[G][v] = [i]$$

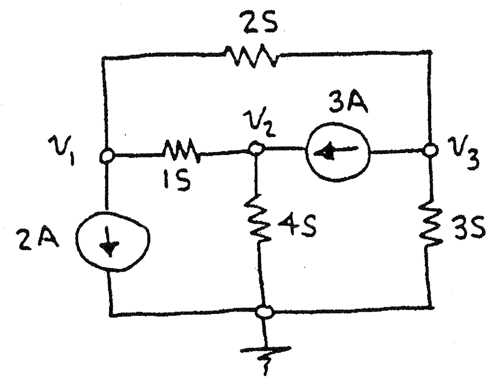
$$[G]^{-1}[G][v] = [G]^{-1}[i]$$

$$[I][v] = [G]^{-1}[i]$$

$$[v] = [G]^{-1}[i]$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{3}{11} \\ \frac{3}{11} & \frac{4}{11} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} \\ -\frac{5}{11} \end{bmatrix}$$

<u>Example</u>:



Because the circuit contains only conductances (or resistances) and independent current sources, the G matrix can be written by inspection.

$$[G] = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 5 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$

$$[i] = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$$

The solution using Cramer's rule is

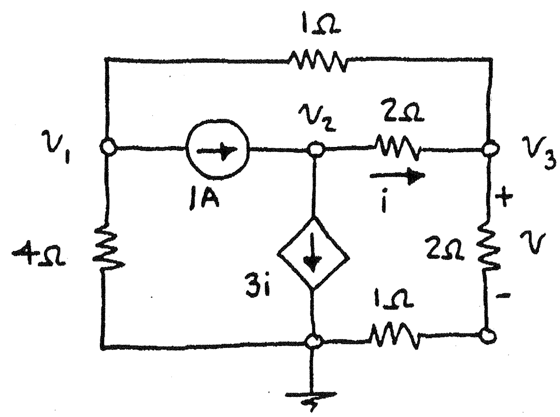
$$v_1 = \frac{\begin{vmatrix} -2 & -1 & -2 \\ 3 & 5 & 0 \\ -3 & 0 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -2 \\ -1 & 5 & 0 \\ -2 & 0 & 5 \end{vmatrix}} = \frac{-50-30+15}{75-20-5} = \frac{-65}{50} = \underline{\underline{-1.30V}}$$

$$v_2 = \frac{\begin{vmatrix} 3 & -2 & -2 \\ -1 & 3 & 0 \\ -2 & -3 & 5 \end{vmatrix}}{50} = \frac{45-6-12-10}{50} = \frac{17}{50} = \underline{\underline{0.34V}}$$

$$v_3 = \frac{\begin{vmatrix} 3 & -1 & -2 \\ -1 & 5 & 3 \\ -2 & 0 & -3 \end{vmatrix}}{50} = \frac{-45+6-20+3}{50} = \frac{-56}{50} = \underline{\underline{-1.12V}}$$

Nodal Analysis With Dependent Sources

Example:



Using KCL at node 1,

$$\frac{v_1}{4} + \frac{v_1 - v_3}{1} + 1 = 0$$

$$v_1 + 4v_1 - 4v_3 + 4 = 0$$

$$5v_1 \qquad -4v_3 = -4$$

At node 2,

$$\begin{cases} -1 + 3i + \dfrac{v_2 - v_3}{2} = 0 \\ i = \dfrac{v_2 - v_3}{2} \end{cases}$$

$$-1 + 3\left(\frac{v_2 - v_3}{2}\right) + \frac{v_2 - v_3}{2} = 0$$

$$-2 + 3v_2 - 3v_3 + v_2 - v_3 = 0$$

$$4v_2 - 4v_3 = 2$$

At node 3,

$$\frac{v_3 - v_2}{2} + \frac{v_3 - v_1}{1} + \frac{v_3}{3} = 0$$

$$3v_3 - 3v_2 + 6v_3 - 6v_1 + 2v_3 = 0$$

$$-6v_1 - 3v_2 + 11v_3 = 0$$

the matrix equation is

$$\begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & -4 \\ -6 & -3 & 11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}$$

therefore,

$$v_1 = \frac{\begin{vmatrix} -4 & 0 & -4 \\ 2 & 4 & -4 \\ 0 & -3 & 11 \end{vmatrix}}{\begin{vmatrix} 5 & 0 & -4 \\ 0 & 4 & -4 \\ -6 & -3 & 11 \end{vmatrix}} = \frac{-176+24+48}{220-96-60} = \frac{-104}{64} = \underline{\underline{-\frac{52}{32}}}\text{ V} \leftarrow$$

$$v_2 = \frac{\begin{vmatrix} 5 & -4 & -4 \\ 0 & 2 & -4 \\ -6 & 0 & 11 \end{vmatrix}}{64} = \frac{110-96-48}{64} = \frac{-34}{64} = \underline{\underline{-\frac{17}{32}}}\text{ V} \leftarrow$$

$$v_3 = \frac{\begin{vmatrix} 5 & 0 & -4 \\ 0 & 4 & 2 \\ -6 & -3 & 0 \end{vmatrix}}{64} = \frac{-96+30}{64} = \frac{-66}{64} = \underline{\underline{-\frac{33}{32}}}\text{ V} \leftarrow$$

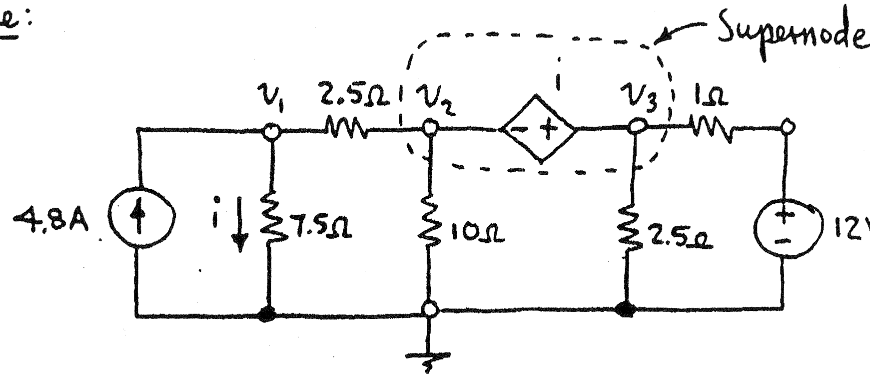
Also,

$$i = \frac{1}{2}(v_2 - v_3) = \frac{1}{2}\left[-\frac{17}{32} - \left(-\frac{33}{32}\right)\right] = \frac{1}{2}\left(\frac{16}{32}\right) = \underline{\underline{\frac{1}{4}}}\text{ A} \leftarrow$$

$$v = \frac{2}{1+2} v_3 = \frac{2}{3}\left(-\frac{33}{32}\right) = \underline{\underline{-\frac{11}{16}}}\text{ V} \leftarrow$$

Nodal Analysis With Voltage Sources

Example:



At node 1,

$$-4.8 + \frac{v_1}{7.5} + \frac{v_1 - v_2}{2.5} = 0$$

$$-36 + v_1 + 3v_1 - 3v_2 = 0$$

$$4v_1 - 3v_2 = 36$$

At the supernode,

$$\frac{v_2 - v_1}{2.5} + \frac{v_2}{10} + \frac{v_3}{2.5} + \frac{v_3 - 12}{1} = 0$$

$$4v_2 - 4v_1 + v_2 + 4v_3 + 10v_3 - 120 = 0$$

$$-4v_1 + 5v_2 + 14v_3 = 120$$

The third equation is

$$v_3 - v_2 = i = \frac{v_1}{7.5}$$

$$15v_3 - 15v_2 = 2v_1$$

$$2v_1 + 15v_2 - 15v_3 = 0$$

The matrix equation is

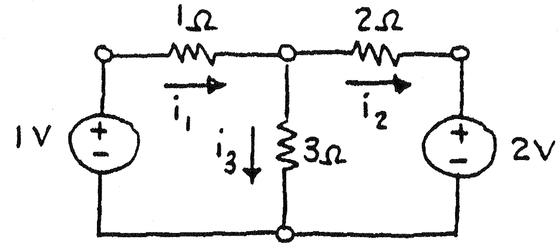
$$\begin{bmatrix} 4 & -3 & 0 \\ -4 & 5 & 14 \\ 2 & 15 & -15 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 36 \\ 120 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 2.7299E-01 & 4.3103E-02 & 4.0230E-02 \\ 3.0651E-02 & 5.7471E-02 & 5.3640E-02 \\ 6.7050E-02 & 6.3218E-02 & -7.6628E-03 \end{bmatrix} \begin{bmatrix} 36 \\ 120 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 10 \end{bmatrix} \longleftarrow$$

Mesh Analysis

Consider the following circuit with branch currents i_1, i_2, and i_3.



Using KVL around the two meshes,

$$1 i_1 + 3 i_3 = 1$$
$$2 i_2 - 3 i_3 = -2$$

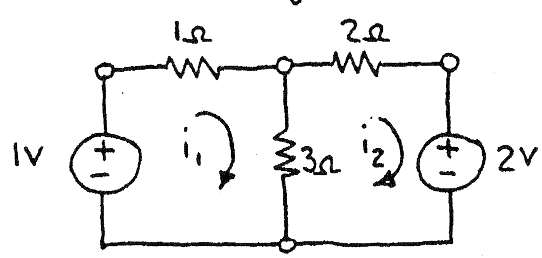
But KCL yields

$$i_1 = i_2 + i_3$$

Eliminating i_3 and gathering terms,

$$\begin{cases} 4 i_1 - 3 i_2 = 1 \\ -3 i_1 + 5 i_2 = -2 \end{cases}$$

These equations are the <u>mesh equations</u> for the above circuit. Redrawing,

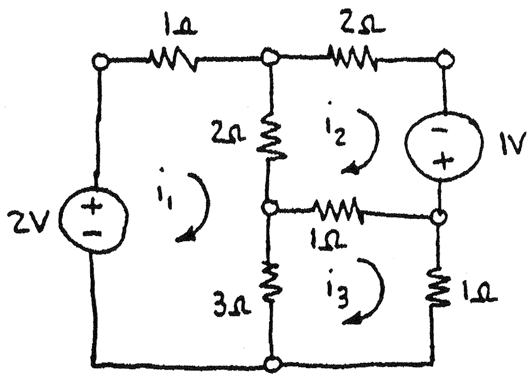


In matrix terms,

$$\begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$[R]\,[i] = [v]$$

Example:



Because the circuit contains only resistances and independent voltage sources, the R matrix can be written by inspection.

$$[R] = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The solution using Cramer's rule is

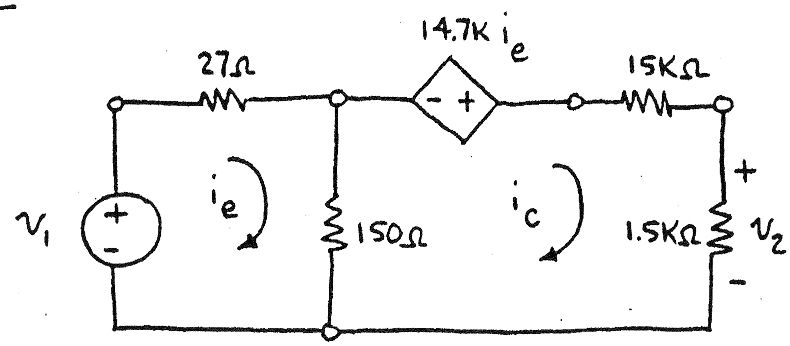
$$i_1 = \frac{\begin{vmatrix} 2 & -2 & -3 \\ 1 & 5 & -1 \\ 0 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{vmatrix}} = \frac{50 + 3 - 2 + 10}{150 - 6 - 6 - 45 - 6 - 20} = \frac{61}{67} \text{A} \leftarrow$$

$$i_2 = \frac{\begin{vmatrix} 6 & 2 & -3 \\ -2 & 1 & -1 \\ -3 & 0 & 5 \end{vmatrix}}{67} = \frac{30 + 6 - 9 + 20}{67} = \frac{47}{67} \text{A} \leftarrow$$

$$i_3 = \frac{\begin{vmatrix} 6 & -2 & 2 \\ -2 & 5 & 1 \\ -3 & -1 & 0 \end{vmatrix}}{67} = \frac{6 + 4 + 30 + 6}{67} = \frac{46}{67} \text{A} \leftarrow$$

Mesh Analysis With Dependent Sources

Example:



Find: $\frac{v_2}{v_1}$ and $\frac{v_1}{i_e}$.

Around loop i_e,

$$27\,i_e + 150(i_e - i_c) - v_1 = 0$$

$$177\,i_e = 150\,i_c + v_1$$

$$i_e = \frac{150\,i_c + v_1}{177}$$

Around loop i_c,

$$-14{,}700\,i_e + 16{,}500\,i_c + 150(i_c - i_e) = 0$$

$$-14{,}850\,i_e + 16{,}650\,i_c = 0$$

$$-14{,}850\left(\frac{150\,i_c + v_1}{177}\right) + 16{,}650\,i_c = 0$$

$$-14{,}850(150\,i_c) - 14{,}850\,v_1 + 16{,}650(177)\,i_c = 0$$

$$719{,}550\,i_c = 14{,}850\,v_1$$

$$i_c = 2.06\times10^{-2}\,v_1$$

Now,

$$v_2 = 1{,}500\, i_c$$

$$= 1{,}500\left(2.06\times10^{-2} v_1\right)$$

$$= 30.90\, v_1$$

$$\therefore \quad \frac{v_2}{v_1} = \underline{\underline{30.90}} \longleftarrow$$

Also,

$$i_e = \frac{150}{177} i_c + \frac{1}{177} v_1$$

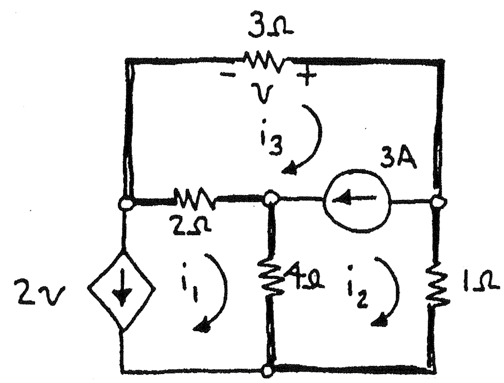
$$= \frac{150}{177}\left(2.06\times10^{-2}\right) v_1 + \frac{1}{177} v_1$$

$$= 2.31\times10^{-2} v_1$$

$$\therefore \quad R_{in} = \frac{v_1}{i_e} = \underline{\underline{43.28\,\Omega}} \longleftarrow$$

Mesh Analysis With Current Sources

Example:



Around loop 1,

$$i_1 = -2v = -2(-3i_3) = 6i_3$$

$$i_1 - 6i_3 = 0$$

Around the supermesh loop,

$$3i_3 + 1i_2 + 4(i_2 - i_1) + 2(i_3 - i_1) = 0$$

$$-6i_1 + 5i_2 + 5i_3 = 0$$

The third equation is

$$i_3 - i_2 = 3$$

The matrix equation is

$$\begin{bmatrix} 1 & 0 & -6 \\ -6 & 5 & 5 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Therefore,

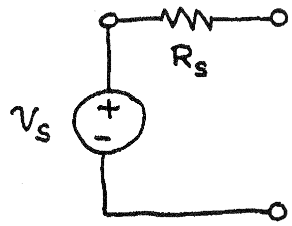
$$i_3 = \frac{\begin{vmatrix} 1 & 0 & 0 \\ -6 & 5 & 0 \\ 0 & -1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & -6 \\ -6 & 5 & 5 \\ 0 & -1 & 1 \end{vmatrix}} = \frac{15}{5 - 36 + 5} = \underline{\underline{-\frac{15}{26}}}\ \text{A} \leftarrow$$

$$i_1 = 6i_3 = 6\left(-\frac{15}{26}\right) = \underline{\underline{-\frac{90}{26}}}\ \text{A} \leftarrow$$

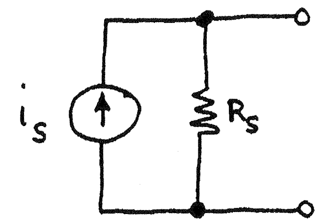
$$i_2 = i_3 - 3 = -\frac{15}{26} - \frac{78}{26} = \underline{\underline{-\frac{93}{26}}}\ \text{A} \leftarrow$$

Nonideal Sources

Nonideal sources are <u>modeled</u> from ideal sources and resistors.

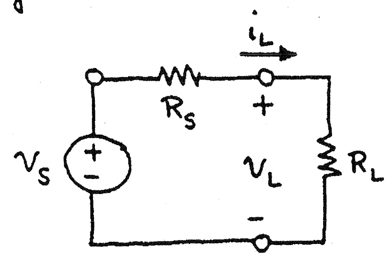


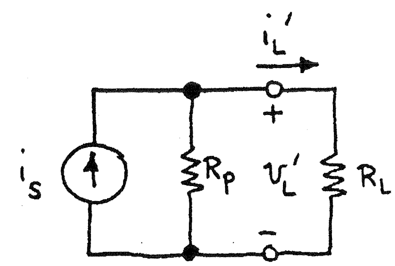
Practical Voltage Source



Practical Current Source

Practical sources are <u>equivalent</u> if they produce the same effect in an arbitrary load.





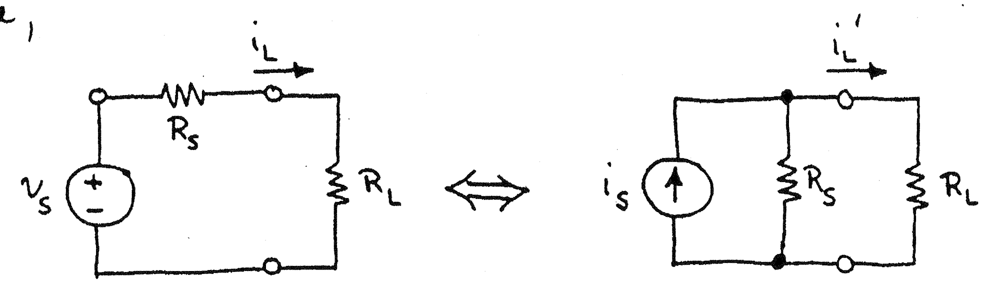
Setting $i_L = i_L'$,

$$\frac{v_s}{R_s + R_L} = \frac{R_p i_s}{R_p + R_L}$$

this equality is satisfied when

$$\boxed{v_s = i_s R_s} \qquad \boxed{R_s = R_p}$$

Therefore,

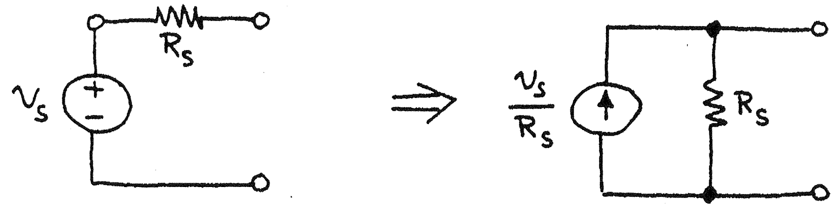


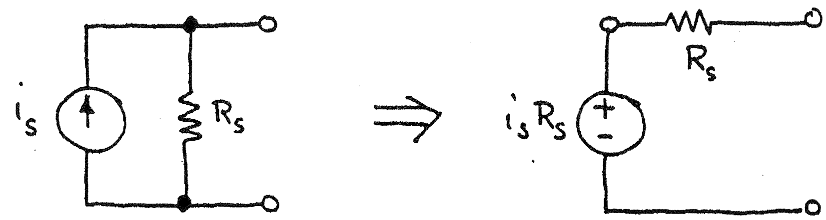
and

$$i_L = \frac{v_s}{R_s + R_L} = \frac{i_s R_s}{R_s + R_L} = i_L'$$

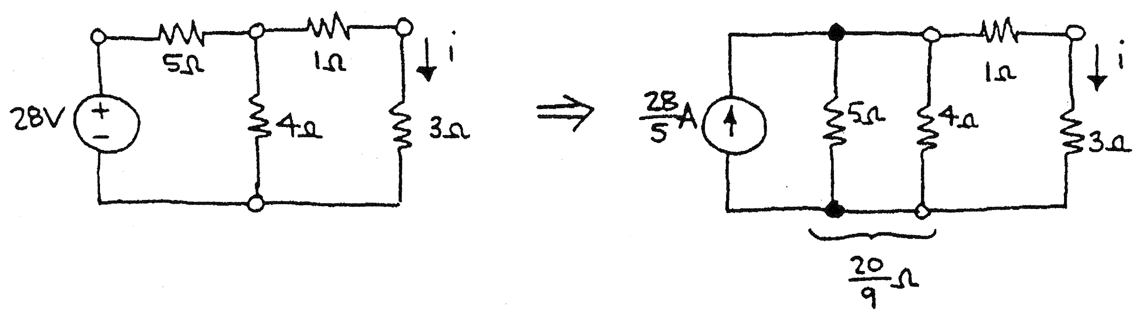
Source Transformations

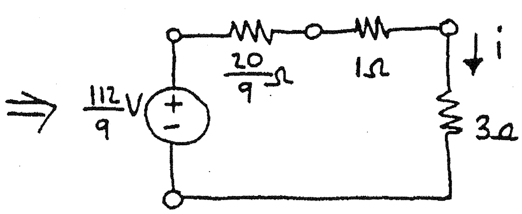
Replacing a source by an equivalent source is called a source transformation. The two kinds of transformations are





Example:



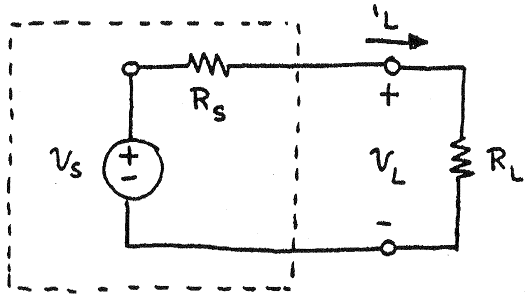


Now,

$$i = \frac{v_s}{R_{eq}} = \frac{\frac{112}{9}}{\frac{20}{9}+1+3} = \frac{\frac{112}{9}}{\frac{56}{9}} = \underline{\underline{2A}}$$

Maximum Power Transfer

Consider the following practical voltage source connected to a load resistance.



Using Ohm's law,

$$i_L = \frac{v_s}{R_s + R_L}$$

the instantaneous power absorbed by the load is

$$p = i_L^2 R_L$$

$$= \left(\frac{v_s}{R_s + R_L}\right)^2 R_L$$

$$= \frac{v_s^2 R_L}{(R_s + R_L)^2}$$

Differentiating,

$$\frac{dp}{dR_L} = \frac{(R_s - R_L)\, v_s^2}{(R_s + R_L)^3} = 0$$

therefore, maximum power occurs when

$$R_s = R_L$$

and is

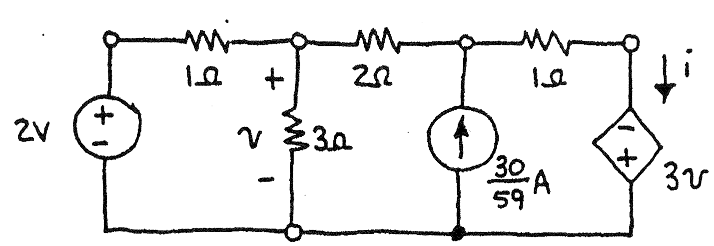
$$\boxed{P_{max} = \frac{v_s^2}{4R_s}}$$

The Superposition Principle

The voltage or current response of a linear circuit containing n independent sources is the sum of the response of each of these n sources considered one at a time with the remaining $n-1$ independent sources set to zero.

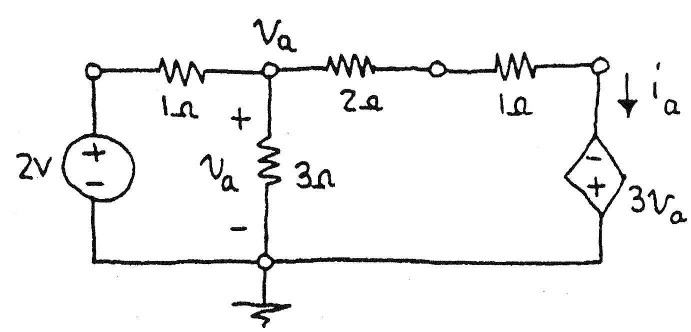
Example:

Consider the following circuit with $n = 2$ independent sources.



Determine i and v using the principle of superposition.

The response to the 2V source is



At node v_a,

$$\frac{v_a - 2}{1} + \frac{v_a}{3} + \frac{v_a - (-3v_a)}{3} = 0$$

$$3v_a - 6 + v_a + v_a + 3v_a = 0$$

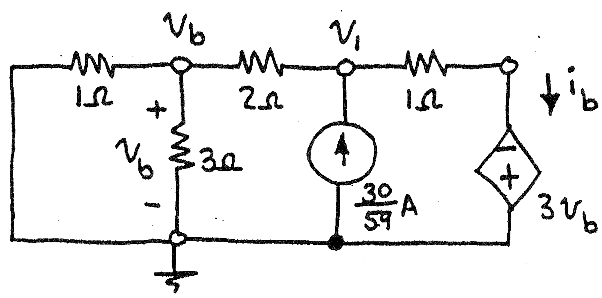
$$8v_a = 6$$

$$v_a = \frac{3}{4}\text{ V}$$

Using Ohm's law,

$$i_a = \frac{v_a - (-3v_a)}{3} = \frac{4}{3}v_a = \frac{4}{3}\left(\frac{3}{4}\right) = 1\text{ A}$$

The response to the $\frac{30}{59}$ A source is



At node v_b,

$$\frac{v_b}{1} + \frac{v_b}{3} + \frac{v_b - v_1}{2} = 0$$

$$6v_b + 2v_b + 3v_b - 3v_1 = 0$$

$$11v_b - 3v_1 = 0$$

At node v_1,

$$\frac{v_1 - v_b}{2} + \frac{v_1 - (-3v_b)}{1} = \frac{30}{59}$$

$$v_1 - v_b + 2v_1 + 6v_b = \frac{60}{59}$$

$$5v_b + 3v_1 = \frac{60}{59}$$

Solving,

$$v_b = \frac{\begin{vmatrix} 0 & -3 \\ \frac{60}{59} & 3 \end{vmatrix}}{\begin{vmatrix} 11 & -3 \\ 5 & 3 \end{vmatrix}} = \frac{\frac{180}{59}}{33 + 15} = \frac{180}{(48)(59)} = \frac{15}{236}\text{ V}$$

$$v_1 = \frac{\begin{vmatrix} 11 & 0 \\ 5 & \frac{60}{59} \end{vmatrix}}{48} = \frac{\frac{660}{59}}{48} = \frac{660}{(48)(59)} = \frac{55}{236} \text{ V}$$

Using Ohm's law,

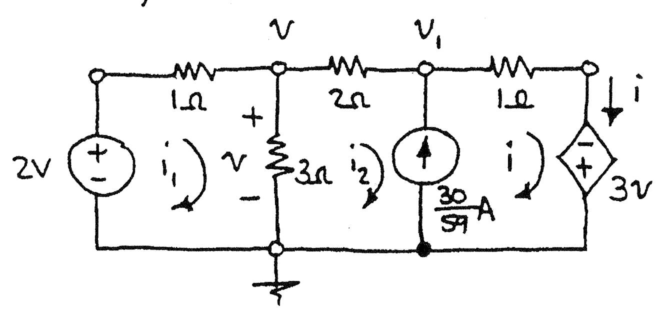
$$i_b = \frac{v_1 - (-3v_b)}{1} = \frac{55}{236} + 3\left(\frac{15}{236}\right) = \frac{100}{236} \text{ A}$$

Summing the responses,

$$i = i_a + i_b = 1 + \frac{100}{236} = \frac{236}{236} + \frac{100}{236} = \frac{336}{236} = \underline{\underline{\frac{84}{59}}} \text{ A}$$

$$v = v_a + v_b = \frac{3}{4} + \frac{15}{236} = \frac{177}{236} + \frac{15}{236} = \frac{192}{236} = \underline{\underline{\frac{48}{59}}} \text{ V}$$

As a check,



Loop 1,

$$1i_1 + 3(i_1 - i_2) = 2$$
$$4i_1 - 3i_2 = 2$$

Supermesh loop,

$$3(i_2 - i_1) + 2i_2 + 1i = 3v = 9(i_1 - i_2)$$
$$-12i_1 + 14i_2 + i = 0$$

By inspection,

$$i_2 - i = -\frac{30}{59}$$

And,

$$i = \frac{\begin{vmatrix} 4 & -3 & 2 \\ -12 & 14 & 0 \\ 0 & 1 & -\frac{30}{59} \end{vmatrix}}{\begin{vmatrix} 4 & -3 & 0 \\ -12 & 14 & 1 \\ 0 & 1 & -1 \end{vmatrix}} = \frac{-56\left(\frac{30}{59}\right) - 24 + 36\left(\frac{30}{59}\right)}{-56 + 36 - 4}$$

$$= \frac{\frac{-1680 - 1416 + 1080}{59}}{-24} = \frac{2016}{(24)(59)} = \underline{\underline{\frac{84}{59}}}\,A \longleftarrow$$

Node v,

$$\frac{v-2}{1} + \frac{v}{3} + \frac{v - v_1}{2} = 0$$

$$6v - 12 + 2v + 3v - 3v_1 = 0$$

$$11v - 3v_1 = 12$$

Node v_1,

$$\frac{v_1 - v}{2} + \frac{v_1 - (-3v)}{1} = \frac{30}{59}$$

$$59v_1 - 59v + 118v_1 + 354v = 60$$

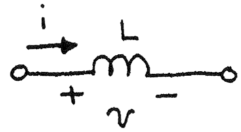
$$295v + 177v_1 = 60$$

And,

$$v = \frac{\begin{vmatrix} 12 & -3 \\ 60 & 177 \end{vmatrix}}{\begin{vmatrix} 11 & -3 \\ 295 & 177 \end{vmatrix}} = \frac{2124 + 180}{1947 + 885} = \frac{2304}{2832} = \underline{\underline{\frac{48}{59}}}\,V \longleftarrow$$

The Inductor

An ideal inductor is an energy storage device that is represented by the following symbol.



The relationship between the voltage and current is

$$\boxed{v = L\frac{di}{dt}}$$

where L is the inductance in henrys.

The energy stored in the inductor at time t is found by integrating the instantaneous absorbed power in the inductor. By definition,

$$\frac{dw_L}{dt} = p$$

$$\therefore \quad w_L = \int_{-\infty}^{t} p\, d\tau = \int_{-\infty}^{t} i v\, d\tau$$

$$= \int_{-\infty}^{t} i\left(L\frac{di}{d\tau}\right) d\tau = L\int_{i(-\infty)}^{i(t)} i\, di$$

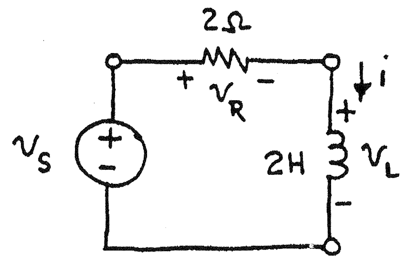
$$= L\frac{i^2}{2}\Bigg|_{i(-\infty)}^{i(t)} = \frac{1}{2}L\left[i^2(t) - i^2(-\infty)\right]$$

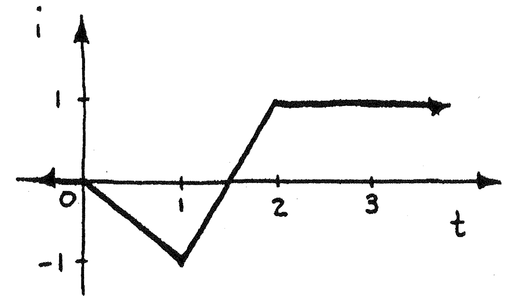
Assuming $i^2(-\infty) = 0$,

$$\boxed{w_L = \frac{1}{2}Li^2}$$

Example :

Consider the following circuit and i waveform.



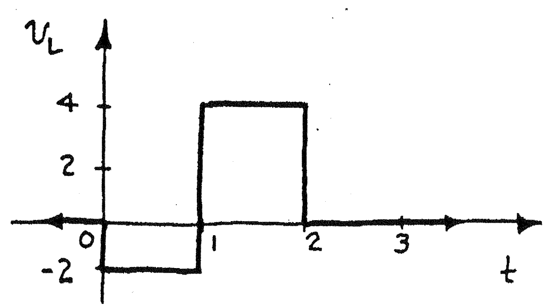


Determine v_L, w_L, P_R, v_R, and v_S.

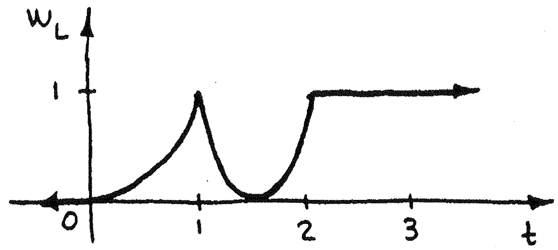
The current waveform can be expressed as

$$i = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ -t & \text{for } 0 \le t < 1 \\ 2t-3 & \text{for } 1 \le t < 2 \\ 1 & \text{for } 2 \le t < \infty \end{cases}$$

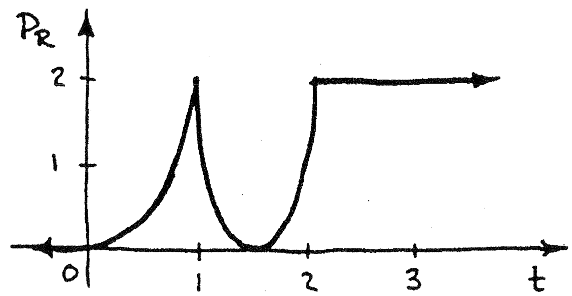
$$v_L = L\frac{di}{dt} = 2\frac{di}{dt} = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ -2 & \text{for } 0 \le t < 1 \\ 4 & \text{for } 1 \le t < 2 \\ 0 & \text{for } 2 \le t < \infty \end{cases}$$



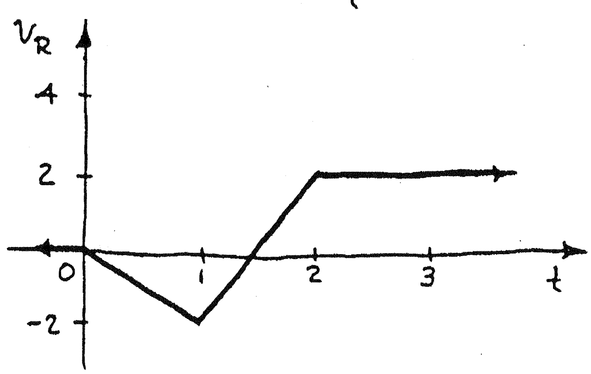
$$w_L = \frac{1}{2}Li^2 = \frac{1}{2}(2)i^2 = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ t^2 & \text{for } 0 \le t < 1 \\ (2t-3)^2 & \text{for } 1 \le t < 2 \\ 1 & \text{for } 2 \le t < \infty \end{cases}$$



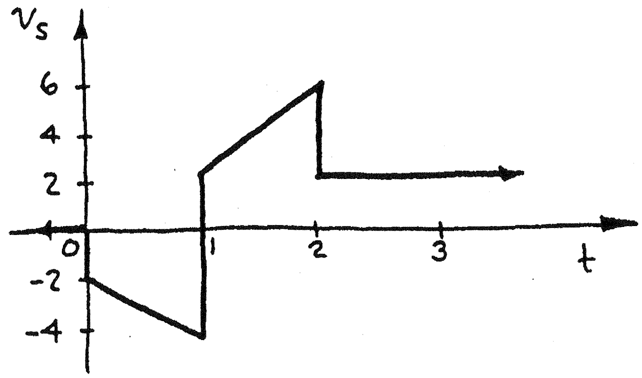
$$P_R = Ri^2 = 2i^2 = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ 2t^2 & \text{for } 0 \le t < 1 \\ 2(2t-3)^2 & \text{for } 1 \le t < 2 \\ 2 & \text{for } 2 \le t < \infty \end{cases}$$



$$v_R = Ri = 2i = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ -2t & \text{for } 0 \le t < 1 \\ 4t-6 & \text{for } 1 \le t < 2 \\ 2 & \text{for } 2 \le t < \infty \end{cases}$$



$$v_S = v_R + v_L = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ -2t-2 & \text{for } 0 \le t < 1 \\ 4t-2 & \text{for } 1 \le t < 2 \\ 2 & \text{for } 2 \le t < \infty \end{cases}$$



The integral relationship between the voltage and current can be derived from the differential relationship.

$$v = L\frac{di}{dt}$$

$$di = \frac{1}{L} v\, dt$$

$$\boxed{i = \frac{1}{L}\int_{-\infty}^{t} v\, d\tau}$$

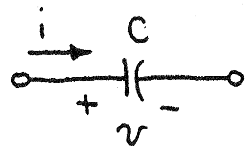
If $i(t)$ is known at $t = t_0$,

$$i(t) = \frac{1}{L}\int_{-\infty}^{t_0} v\, d\tau + \frac{1}{L}\int_{t_0}^{t} v\, d\tau$$

$$\boxed{i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v\, d\tau}$$

The Capacitor

An ideal capacitor is an energy storage device that is represented by the following symbol.



The relationship between the voltage and current is

$$\boxed{i = C\frac{dv}{dt}}$$

where C is the capacitance in <u>farads</u>.

The energy stored in the capacitor at time t is found by integrating the instantaneous absorbed power in the capacitor. By definition,

$$\frac{dw_c}{dt} = p$$

$$\therefore \quad w_c = \int_{-\infty}^{t} p\,d\tau = \int_{-\infty}^{t} vi\,d\tau$$

$$= \int_{-\infty}^{t} v\left(C\frac{dv}{d\tau}\right)d\tau = C\int_{v(-\infty)}^{v(t)} v\,dv$$

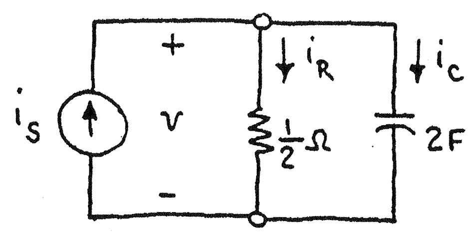
$$= C\frac{v^2}{2}\Bigg|_{v(-\infty)}^{v(t)} = \frac{1}{2}C\left[v^2(t) - v^2(-\infty)\right]$$

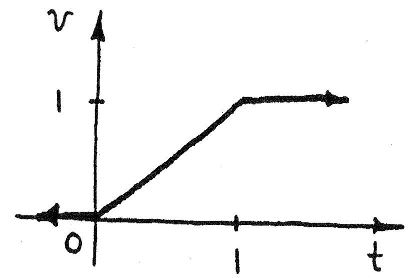
Assuming $v^2(-\infty) = 0$,

$$\boxed{w_c = \frac{1}{2}Cv^2}$$

Example:

Consider the following circuit and v waveform.



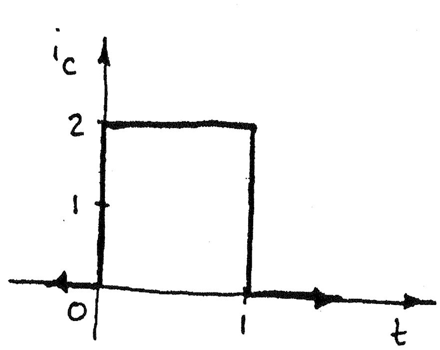


Determine i_c, w_c, P_R, i_R, and i_s.

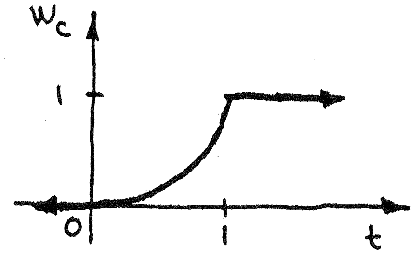
The voltage waveform can be expressed as

$$v = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ t & \text{for } 0 \le t < 1 \\ 1 & \text{for } 1 \le t < \infty \end{cases}$$

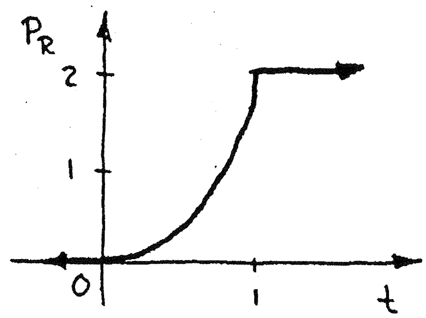
$$i_c = C\frac{dv}{dt} = 2\frac{dv}{dt} = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ 2 & \text{for } 0 \le t < 1 \\ 0 & \text{for } 1 \le t < \infty \end{cases}$$



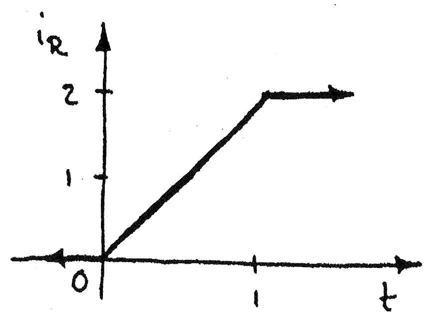
$$w_c = \frac{1}{2} C v^2 = \frac{1}{2}(2)\, v^2 = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ t^2 & \text{for } 0 \leq t < 1 \\ 1 & \text{for } 1 \leq t < \infty \end{cases}$$



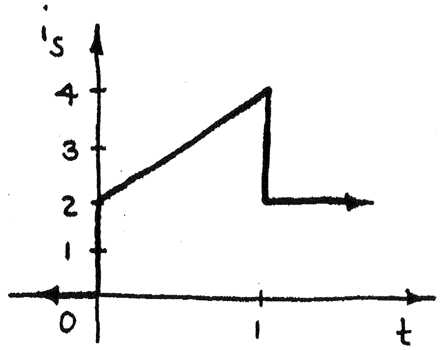
$$P_R = \frac{v^2}{R} = 2v^2 = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ 2t^2 & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < \infty \end{cases}$$



$$i_R = \frac{v}{R} = 2v = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ 2t & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < \infty \end{cases}$$



$$i_s = i_R + i_C = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ 2t+2 & \text{for } 0 \le t < 1 \\ 2 & \text{for } 1 \le t < \infty \end{cases}$$



The integral relationship between the voltage and current can be derived from the differential relationship.

$$i = C\frac{dv}{dt}$$

$$dv = \frac{1}{C}\,i\,dt$$

$$\boxed{v = \frac{1}{C}\int_{-\infty}^{t} i\,d\tau}$$

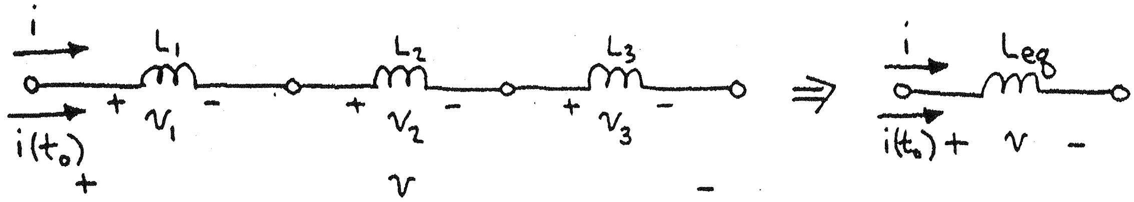
If $v(t)$ is known at $t = t_0$,

$$v(t) = \frac{1}{C}\int_{-\infty}^{t_0} i\,d\tau + \frac{1}{C}\int_{t_0}^{t} i\,d\tau$$

$$\boxed{v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i\,d\tau}$$

Series - Parallel Combination of Inductance and Capacitance

Inductors in series:



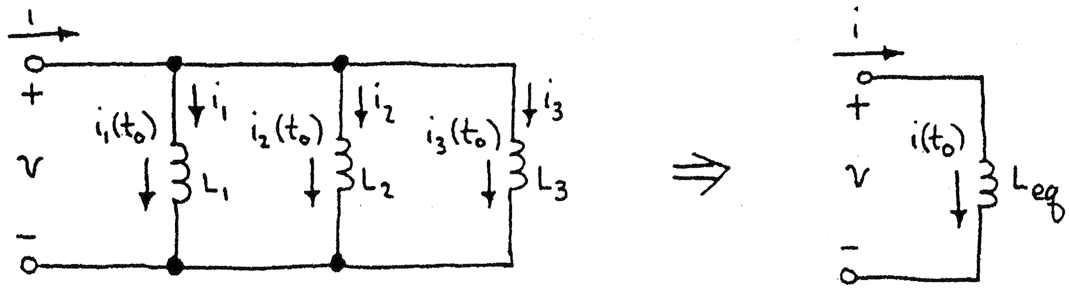
Using KVL,

$$v = v_1 + v_2 + v_3 = L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt} = (L_1+L_2+L_3)\frac{di}{dt} = L_{eq}\frac{di}{dt}$$

In general,

$$\boxed{L_{eq} = \sum_{i=1}^{K} L_i = L_1 + L_2 + \cdots + L_K}$$

Inductors in parallel:



Using KCL,

$$i = i_1 + i_2 + i_3 = \frac{1}{L_1}\int_{t_0}^{t} v\,d\tau + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,d\tau + i_2(t_0) + \frac{1}{L_3}\int_{t_0}^{t} v\,d\tau + i_3(t_0)$$

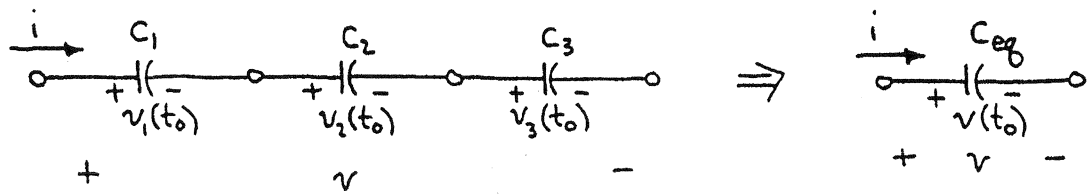
$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int_{t_0}^{t} v\,d\tau + i_1(t_0) + i_2(t_0) + i_3(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} v\,d\tau + i(t_0)$$

In general,

$$\boxed{\frac{1}{L_{eq}} = \sum_{i=1}^{K} \frac{1}{L_i} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_K}}$$

$$\boxed{i(t_0) = \sum_{i=1}^{K} i_K(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_K(t_0)}$$

Capacitors in series:

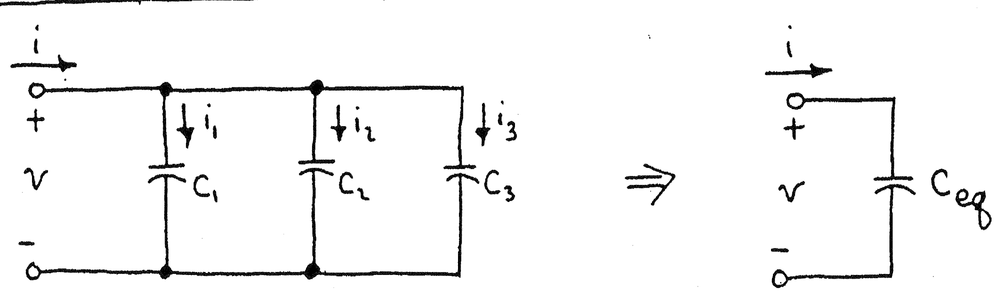


Applying KVL and generalizing,

$$\boxed{\frac{1}{C_{eq}} = \sum_{i=1}^{K} \frac{1}{C_i} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_K}}$$

$$\boxed{v(t_0) = \sum_{i=1}^{K} v_K(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_K(t_0)}$$

Capacitors in parallel:

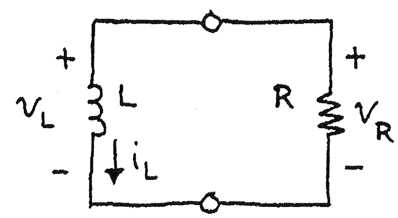


Applying KCL and generalizing,

$$\boxed{C_{eq} = \sum_{i=1}^{K} C_i = C_1 + C_2 + \cdots + C_K}$$

The Natural Response of an RL Circuit

Consider the following RL circuit where i_L at time $t = 0$ is known.



By KVL,

$$v_L - v_R = 0$$

$$L\frac{di_L}{dt} - (-Ri_L) = 0$$

$$\frac{di_L}{dt} + \frac{R}{L}i_L = 0$$

this is a <u>homogeneous</u> <u>first-order</u> <u>linear</u> <u>differential equation</u>.

Solving,

$$\frac{di_L}{dt} + \frac{R}{L}i_L = 0$$

$$\frac{di_L}{i_L} = -\frac{R}{L}\,dt$$

$$\int \frac{di_L}{i_L} = -\frac{R}{L}\int dt$$

$$\ln i_L(t) = -\frac{Rt}{L} + K_1$$

$$i_L(t) = e^{-\frac{Rt}{L} + K_1} = e^{-\frac{Rt}{L}} e^{K_1}$$

$$= Ke^{-\frac{Rt}{L}}$$

At $t = 0$,

$$i_L(0) = Ke^0 = K = I_0$$

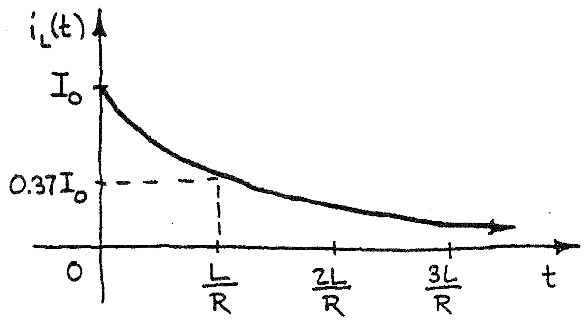
therefore,

$$\boxed{i_L(t) = I_0 e^{\frac{-Rt}{L}}, \quad t \geq 0}$$

the resistor voltage v_R can be determined using Ohm's law.

$$v_R(t) = -R\,i_L(t) = -RI_0 e^{\frac{-Rt}{L}}, \quad t \geq 0$$

The graph of $i_L(t)$ or $v_R(t)$ is the <u>zero-input</u> or <u>natural response</u> of the circuit.



Both these equations have the same form,

$$f(t) = f(0)e^{-\frac{t}{\tau}}$$

where $\tau = \frac{L}{R}$ is called the <u>time constant</u>.

The inductor voltage v_L is

$$v_L(t) = L\frac{di_L(t)}{dt} = L\frac{d}{dt}\left(I_0 e^{\frac{-Rt}{L}}\right) = L\left(-\frac{R}{L}\right)I_0 e^{-\frac{Rt}{L}}$$

$$= -RI_0 e^{-\frac{Rt}{L}}, \quad t \geq 0$$

which is the same as $v_R(t)$.

The initial energy stored in the inductor at $t=0$ is

$$w_L(0) = \frac{1}{2} L I_0^2$$

The power absorbed by the resistor is

$$p_R(t) = R i_L^2(t) = R\left(I_0 e^{-\frac{Rt}{L}}\right)^2 = R I_0^2 e^{-\frac{2Rt}{L}}$$

The total energy absorbed by the resistor is

$$w_R(t) = \int_0^\infty p_R(t)\, dt = \int_0^\infty R I_0^2 e^{-\frac{2Rt}{L}} dt$$

$$= -\frac{L}{2R} R I_0^2 e^{-\frac{2Rt}{L}} \Big|_0^\infty = -\frac{L}{2} I_0^2 (0-1)$$

$$= \frac{1}{2} L I_0^2 = w_L(0)$$

Now consider the RL circuit where i_L at time $t=t_0$ is known. Starting with

$$i_L(t) = K e^{-\frac{Rt}{L}}$$

and evaluating K at $t=t_0$,

$$i_L(t_0) = K e^{-\frac{Rt_0}{L}}$$

$$\therefore \quad K = i_L(t_0) e^{\frac{Rt_0}{L}}$$

Substituting,

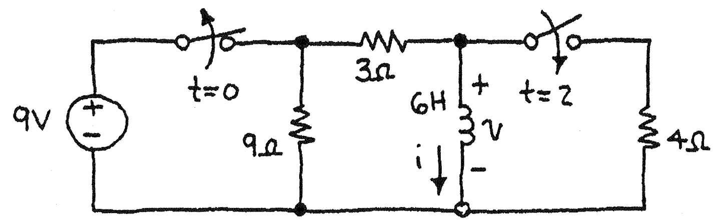
$$i_L(t) = i_L(t_0) e^{-\frac{Rt}{L}} e^{\frac{Rt_0}{L}} = i_L(t_0) e^{\frac{-Rt+Rt_0}{L}}$$

$$\boxed{i_L(t) = i_L(t_0) e^{\frac{-R(t-t_0)}{L}}, \quad t \geq t_0}$$

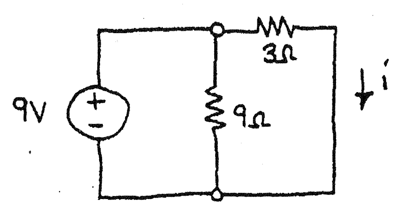
This is a more general expression, and $t_0 = 0$ is merely a special case.

Example:

Find $i(t)$ and $v(t)$ for all time and sketch these functions.



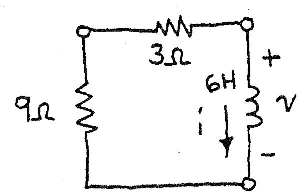
For $t<0$,



$$v(t) = 0\,V$$

$$i(t) = \frac{9}{3} = 3A \Rightarrow I_o = 3A$$

For $0 \le t < 2$,



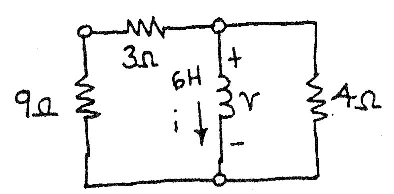
$$i(t) = I_o e^{\frac{-Rt}{L}} = 3e^{-2t}$$

$$v(t) = L\frac{di(t)}{dt} = 6\frac{d}{dt}\left(3e^{-2t}\right) = -36e^{-2t}$$

$$i(2) = 3e^{-2(2)} = 0.055\,A$$

$$v(2^-) = -36e^{-2(2)} = -0.66\,V$$

For $t \ge 2$,



$$R_{eq} = \frac{(12)(4)}{12+4} = 3\Omega$$

$$i(t) = i(t_o)e^{\frac{-R(t-t_o)}{L}} = i(2)e^{\frac{-R(t-2)}{L}} = 0.055\,e^{\frac{-(t-2)}{2}}$$

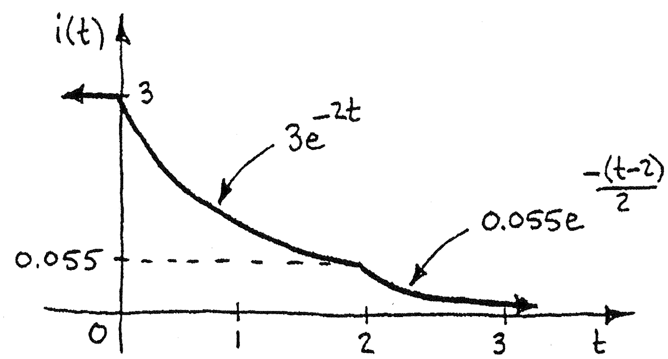
$$v(t) = L\frac{di(t)}{dt} = 6\frac{d}{dt}\left[0.055\,e^{\frac{-(t-2)}{2}}\right] = -0.165e^{\frac{-(t-2)}{2}}$$

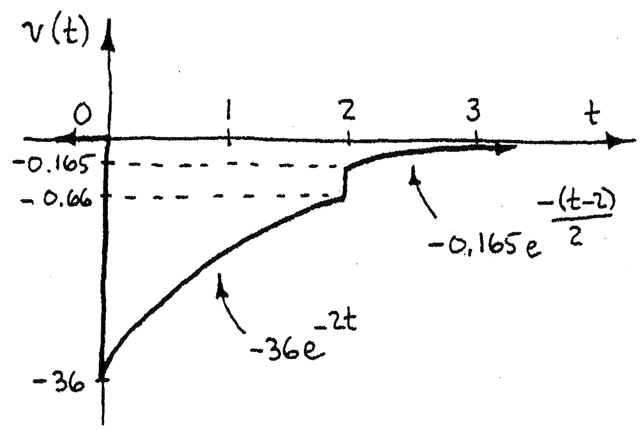
For all t,

$$i(t) = \begin{cases} 3 & \text{for } t < 0 \\ 3e^{-2t} & \text{for } 0 \le t < 2 \\ 0.055e^{\frac{-(t-2)}{2}} & \text{for } t \ge 2 \end{cases}$$

$$v(t) = \begin{cases} 0 & \text{for } t < 0 \\ -36e^{-2t} & \text{for } 0 \le t < 2 \\ -0.165e^{\frac{-(t-2)}{2}} & \text{for } t \ge 2 \end{cases}$$

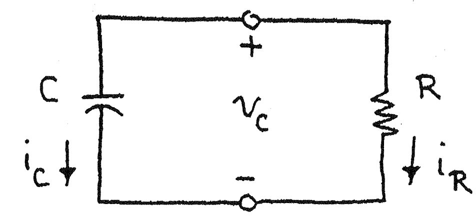
Sketching,





The Natural Response of an RC Circuit

Consider the following RC circuit where v_c at time $t = 0$ is known.



By KCL,

$$i_c + i_R = 0$$

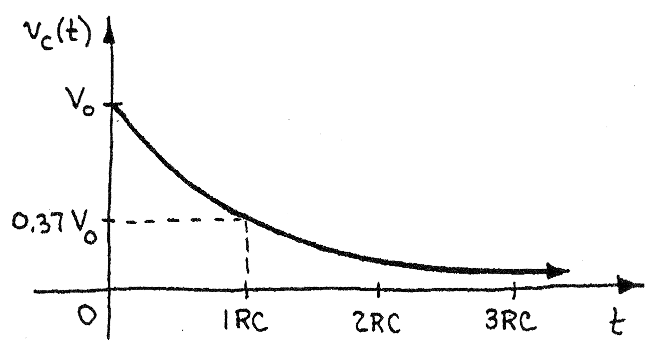
$$C\frac{dv_c}{dt} + \frac{v_c}{R} = 0$$

$$\frac{dv_c}{dt} + \frac{1}{RC}v_c = 0$$

Solving,

$$\boxed{v_c(t) = V_o e^{-\frac{t}{RC}}, \quad t \geq 0}$$

Graphing,



The time constant τ is now RC.

The resistor current i_R is

$$i_R(t) = \frac{v_R(t)}{R} = \frac{v_c(t)}{R} = \frac{V_o}{R}e^{-\frac{t}{RC}}, \quad t \geq 0$$

Also,

$$i_c(t) = -i_R(t) = -\frac{V_o}{R} e^{-\frac{t}{RC}}, \quad t \geq 0$$

or

$$i_c(t) = C\frac{dv_c(t)}{dt} = C\frac{d}{dt}\left(V_o e^{-\frac{t}{RC}}\right) = C\left(-\frac{1}{RC}\right)V_o e^{-\frac{t}{RC}}$$

$$= -\frac{V_o}{R} e^{-\frac{t}{RC}}, \quad t \geq 0$$

The initial energy stored in the capacitor at $t = 0$ is

$$w_c(0) = \frac{1}{2}CV_o^2$$

The power absorbed by the resistor is

$$P_R(t) = Ri_R^2(t) = R\left(\frac{V_o}{R}e^{-\frac{t}{RC}}\right)^2 = \frac{V_o^2}{R}e^{-\frac{2t}{RC}}$$

The total energy absorbed by the resistor is

$$w_R(t) = \int_0^\infty P_R(t)\,dt = \int_0^\infty \frac{V_o^2}{R}e^{-\frac{2t}{RC}}\,dt$$

$$= -\frac{RC}{2}\frac{V_o^2}{R}e^{-\frac{2t}{RC}}\Bigg|_0^\infty = -\frac{C}{2}V_o^2(0-1)$$

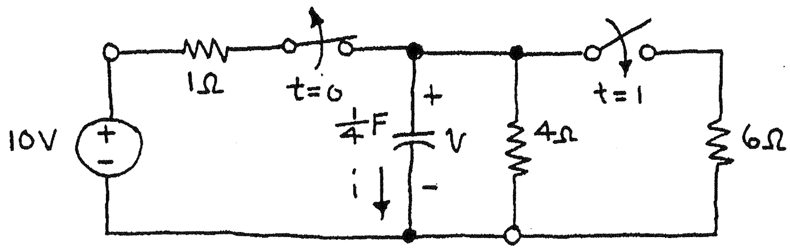
$$= \frac{1}{2}CV_o^2 = w_c(0)$$

If v_c is known at $t = t_o$,

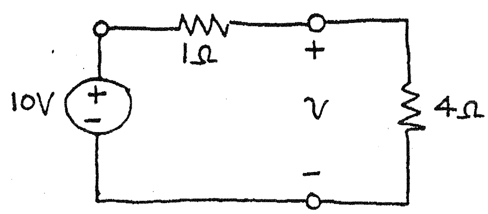
$$\boxed{v_c(t) = v_c(t_o)e^{-\frac{(t-t_o)}{RC}}, \quad t \geq t_o}$$

Example :

Find $v(t)$ and $i(t)$ for all time and sketch these functions. Also, determine the time when $w_c(t) = 0.01\, w_c(0)$.



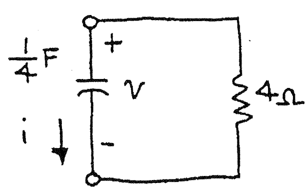
For $t < 0$,



$$i(t) = 0\text{A}$$

$$v(t) = \frac{4}{4+1}(10) = 8\text{V} \Rightarrow V_o = 8\text{V}$$

For $0 \le t < 1$,



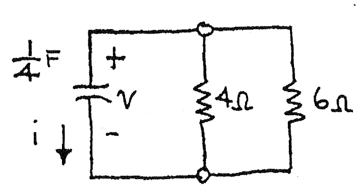
$$v(t) = V_o e^{\frac{-t}{RC}} = 8e^{-t}$$

$$i(t) = C\frac{dv(t)}{dt} = \frac{1}{4}\frac{d}{dt}\left(8e^{-t}\right) = -2e^{-t}$$

$$v(1) = 8e^{-1} = 2.94\text{V}$$

$$i(1^-) = -2e^{-1} = -0.74\text{A}$$

For $t \ge 1$,



$$R_{eq} = \frac{(4)(6)}{4+6} = 2.40\Omega$$

$$v(t) = v(t_o)e^{\frac{-(t-t_o)}{RC}} = v(1)e^{\frac{-(t-1)}{RC}} = 2.94e^{\frac{-5(t-1)}{3}}$$

$$i(t) = C\frac{dv(t)}{dt} = \frac{1}{4}\frac{d}{dt}\left[2.94e^{\frac{-5(t-1)}{3}}\right] = -1.23e^{\frac{-5(t-1)}{3}}$$

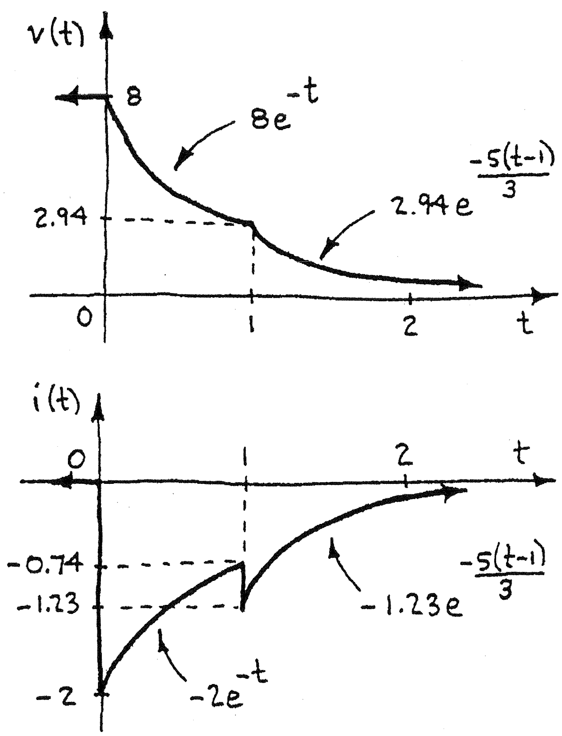
For all t,

$$v(t) = \begin{cases} 8 & \text{for } t < 0 \\ 8e^{-t} & \text{for } 0 \le t < 1 \\ 2.94e^{\frac{-5(t-1)}{3}} & \text{for } t \ge 1 \end{cases}$$

$$i(t) = \begin{cases} 0 & \text{for } t < 0 \\ -2e^{-t} & \text{for } 0 \le t < 1 \\ -1.23e^{\frac{-5(t-1)}{3}} & \text{for } t \ge 1 \end{cases}$$

---

sketching,



For t = 0,

$$w_c(0) = \frac{1}{2}CV_0^2 = \frac{1}{2}\left(\frac{1}{4}\right)(8)^2 = 8\text{J}$$

therefore,

$$w_c(t) = \frac{1}{2} C v^2(t) = 0.01 w_c(0) = 0.01(8) = 0.08 \text{ J}$$

$$v^2(t) = \frac{(2)(0.08)}{C} = 0.64$$

$$v(t) = 0.80 \text{ V}$$

Solving for t,

$$v(t) = 2.94 e^{\frac{-5(t-1)}{3}} = 0.80$$

$$e^{\frac{-5(t-1)}{3}} = \frac{0.80}{2.94} = 0.27$$

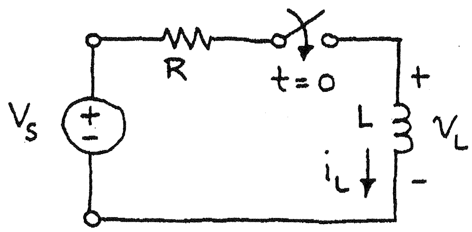
$$\frac{-5(t-1)}{3} = \ln 0.27 = -1.31$$

$$t-1 = \frac{(3)(1.31)}{5} = 0.79$$

$$t = \underline{\underline{1.79 \text{ sec}}} \leftarrow$$

The Step Response of an RL Circuit

Consider the following RL circuit where $i_L(0) = 0$.



Applying KVL at $t \geq 0$,

$$L\frac{di_L}{dt} + Ri_L = V_s$$

$$\frac{di_L}{dt} + \frac{R}{L}i_L = \frac{V_s}{L}$$

The general solution of the homogeneous equation is

$$i_L(\text{general}) = K_1 e^{\frac{-Rt}{L}}$$

For the particular solution, $i_L = K_2$ will be tried.

$$\frac{d}{dt}(K_2) + \frac{RK_2}{L} = \frac{V_s}{L}$$

$$\frac{RK_2}{L} = \frac{V_s}{L}$$

$$K_2 = \frac{V_s}{R}$$

Therefore,

$$i_L(\text{particular}) = \frac{V_s}{R}$$

---

the complete solution is

$$i_L(t) = i_L(\text{particular}) + i_L(\text{general})$$

$$= \frac{V_S}{R} + K_1 e^{-\frac{Rt}{L}}$$

Knowing $i_L(0) = 0$ permits the evaluation of K_1.

$$0 = \frac{V_S}{R} + K_1 e^0$$

$$\therefore \quad K_1 = -\frac{V_S}{R}$$

Finally,

$$\boxed{i_L(t) = \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{Rt}{L}}, \quad t \geq 0}$$

As a check,

$$\frac{di_L}{dt} + \frac{R}{L} i_L = \frac{V_S}{L}$$

$$\frac{d}{dt}\left(\frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{Rt}{L}}\right) + \frac{R}{L}\left(\frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{Rt}{L}}\right) = \frac{V_S}{L}$$

$$\frac{V_S}{L} e^{-\frac{Rt}{L}} + \frac{V_S}{L} - \frac{V_S}{L} e^{-\frac{Rt}{L}} = \frac{V_S}{L}$$

$$\frac{V_S}{L} = \frac{V_S}{L}$$

---

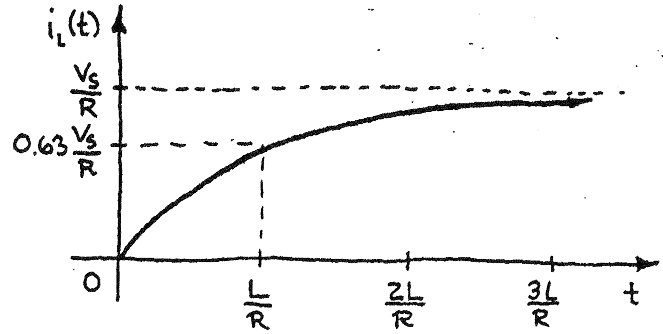
the inductor voltage is

$$v_L(t) = V_S - R i_L(t) = V_S - R\left(\frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{Rt}{L}}\right)$$

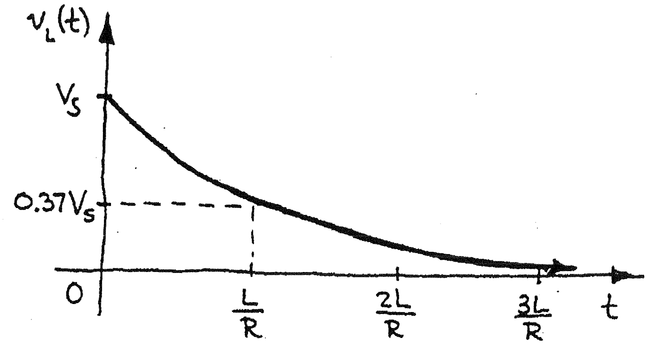
$$= V_S - V_S + V_S e^{-\frac{Rt}{L}}$$

$$= V_S e^{-\frac{Rt}{L}}, \quad t \geq 0$$

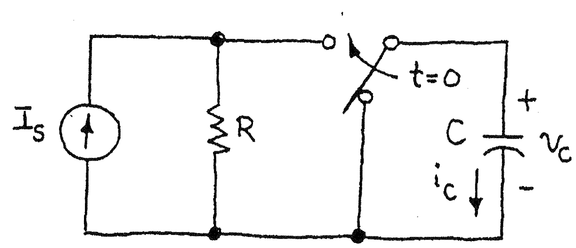
The graph of this particular <u>zero-state</u> response is a <u>step response</u>.



The graph of the voltage is



The following parallel RC circuit



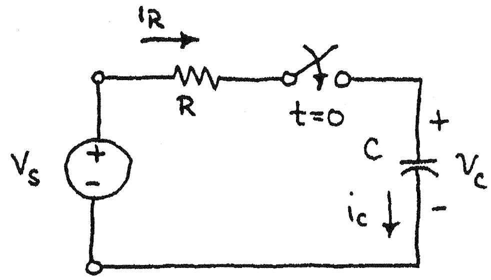
is the <u>dual</u> of the series RL circuit. Therefore,

$$v_c(t) = I_S R - I_S R e^{-\frac{t}{RC}}, \quad t \geq 0$$

$$i_c(t) = I_S e^{-\frac{t}{RC}}, \quad t \geq 0$$

The Step Response of an RC Circuit

Consider the following RC circuit where $v_c(0) = V_o$.



Applying KCL at $t \geq 0$,

$$i_c = i_R$$

$$C\frac{dv_c}{dt} = \frac{V_s - v_c}{R}$$

$$C\frac{dv_c}{dt} + \frac{v_c}{R} = \frac{V_s}{R}$$

$$\frac{dv_c}{dt} + \frac{1}{RC}v_c = \frac{V_s}{RC}$$

The complete solution is

$$v_c(t) = v_c(\text{particular}) + v_c(\text{general})$$

$$= V_s + Ke^{-\frac{t}{RC}}$$

Knowing $v_c(0) = V_o$,

$$V_o = V_s + Ke^0$$

$$\therefore \quad K = V_o - V_s$$

therefore,

$$\boxed{v_c(t) = V_s + (V_o - V_s)e^{-\frac{t}{RC}}, \quad t \geq 0}$$

If $V_0 = 0$,

$$\boxed{v_c(t) = V_s - V_s e^{-\frac{t}{RC}}, \quad t \geq 0}$$

The capacitor current is

$$i_c(t) = \frac{V_s}{R} - \frac{v_c}{R}$$

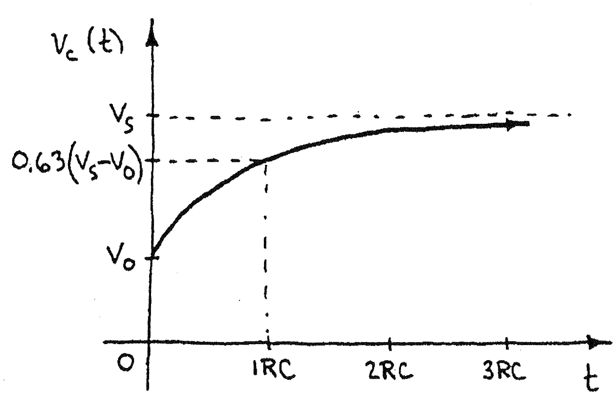
$$= \frac{V_s}{R} - \frac{V_s}{R} - \frac{V_0 - V_s}{R} e^{-\frac{t}{RC}}$$

$$= \frac{V_s - V_0}{R} e^{-\frac{t}{RC}}, \quad t \geq 0$$

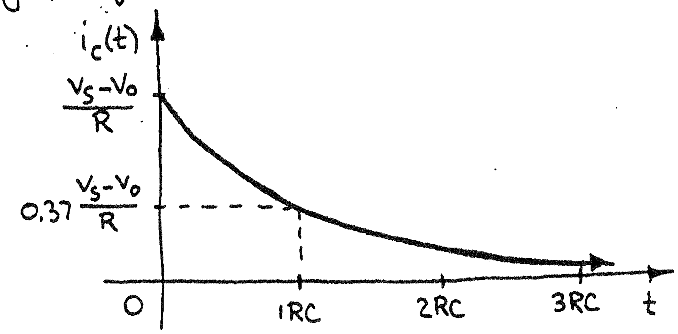
Again, if $V_0 = 0$,

$$i_c(t) = \frac{V_s}{R} e^{-\frac{t}{RC}}, \quad t \geq 0$$

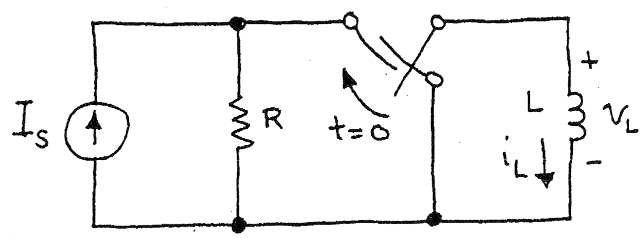
The graph of the voltage is



the graph of the current is



The following parallel RL circuit



is the dual of the series RC circuit. Therefore,

$$i_L(t) = I_S + (I_o - I_S)\,e^{-\frac{Rt}{L}} \quad , \quad t \geq 0$$

$$v_L(t) = (I_S R - I_o R)\,e^{-\frac{Rt}{L}} \quad , \quad t \geq 0$$

For $I_o = 0$,

$$i_L(t) = I_S - I_S\,e^{-\frac{Rt}{L}} \quad , \quad t \geq 0$$

$$v_L(t) = I_S R\,e^{-\frac{Rt}{L}} \quad , \quad t \geq 0$$

Forced and Natural Response

Consider the following general first-order linear differential equation with constant coefficients, where $f(t)$ results from the input and $x(t)$ represents the response.

$$\frac{dx(t)}{dt} + a\,x(t) = f(t)$$

To obtain a general solution, multiply both sides of the equation by e^{at}.

$$e^{at}\frac{dx(t)}{dt} + e^{at}a\,x(t) = e^{at}f(t)$$

the left side of the equation is now an exact derivative.

$$\frac{d}{dt}\left[e^{at}x(t)\right] = e^{at}f(t)$$

$$\int d\left[e^{at}x(t)\right] = \int e^{at}f(t)\,dt$$

$$e^{at}x(t) = \int e^{at}f(t)\,dt + K$$

$$x(t) = e^{-at}\int e^{at}f(t)\,dt + Ke^{-at}$$

In summary, the general solution of the differential equation

$$\frac{dx(t)}{dt} + a\,x(t) = f(t)$$

is called the complete response.

forcing function

$$\underbrace{x(t)}_{\text{complete response}} = \underbrace{e^{-at}\int e^{at}f(t)\,dt}_{\substack{\text{forced response}\\ \text{(steady-state response)}}} + \underbrace{Ke^{-at}}_{\substack{\text{natural response}\\ \text{(transient response)}}}$$

If the forcing function $f(t)$ is a constant, say b, then the differential equation is

$$\frac{dx(t)}{dt} + a\,x(t) = b$$

and the complete response is

$$x(t) = x_f(t) + x_n(t)$$

$$= \frac{b}{a} + Ke^{-at}$$

where the constant K is determined from an initial (boundary) condition. For example, when $t = 0$,

$$x(0) = \frac{b}{a} + Ke^{0}$$

$$\therefore \quad K = x(0) - \frac{b}{a}$$

Substituting,

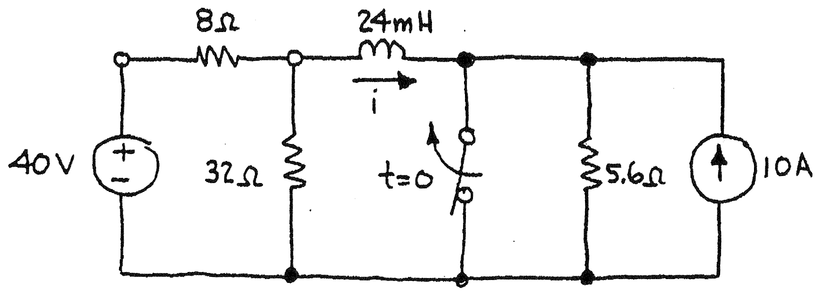
$$x(t) = \frac{b}{a} + \left[x(0) - \frac{b}{a}\right] e^{-at}$$

In general,

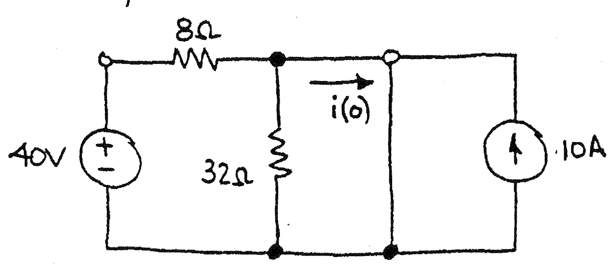
$$\boxed{x(t) = x_f(t) + \left[x(0) - x_f(t)\right] e^{-\frac{t}{\tau}}}$$

Example:

Find $i(t)$ for all time.

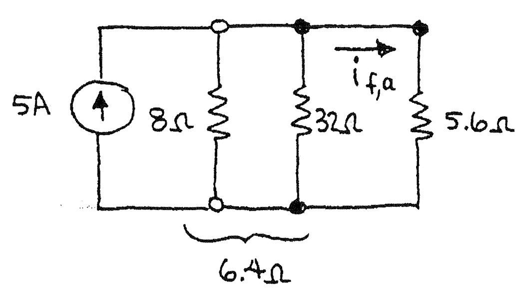


For $t < 0$,



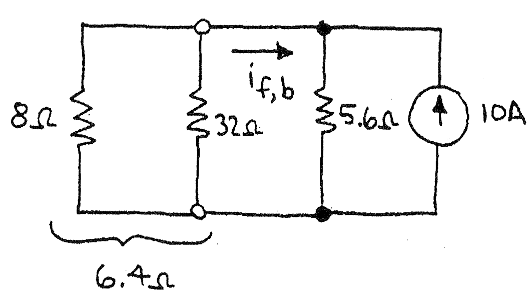
$$i(t) = \frac{40}{8} = 5A \Rightarrow i(0) = 5A$$

Superpositioning the transformed 40V source for $t \geq 0$,



$$i_{f,a} = \frac{6.4}{6.4+5.6}(5) = 2.\overline{66}\,A$$

Superpositioning the 10A source for $t \geq 0$,

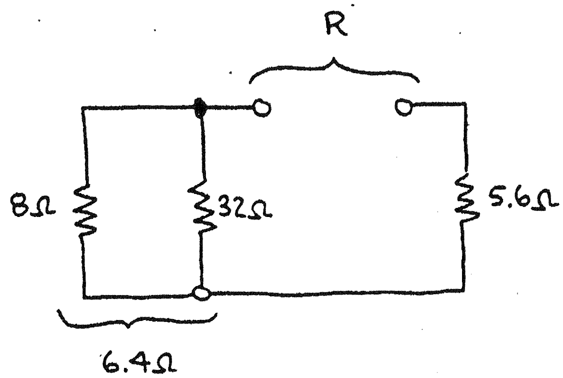


$$i_{f,b} = \frac{5.6}{5.6+6.4}(-10) = -4.\overline{66}\,A$$

Adding,

$$i_f(t) = i_{f,a} + i_{f,b} = 2.\overline{66} + (-4.\overline{66}) = -2A$$

The Thevenized resistance across the inductor terminals for $t \geq 0$ is



$R = 6.4 + 5.6 = 12\,\Omega$

The time constant is

$$\tau = \frac{L}{R} = \frac{24\times10^{-3}}{12} = 2\text{ ms}$$

Using the general equation,

$$i(t) = i_f(t) + \left[i(0) - i_f(t) \right] e^{-\frac{t}{\tau}}$$

$$= -2 + \left[5 - (-2) \right] e^{-\frac{t}{2\times10^{-3}}}$$

Finally,

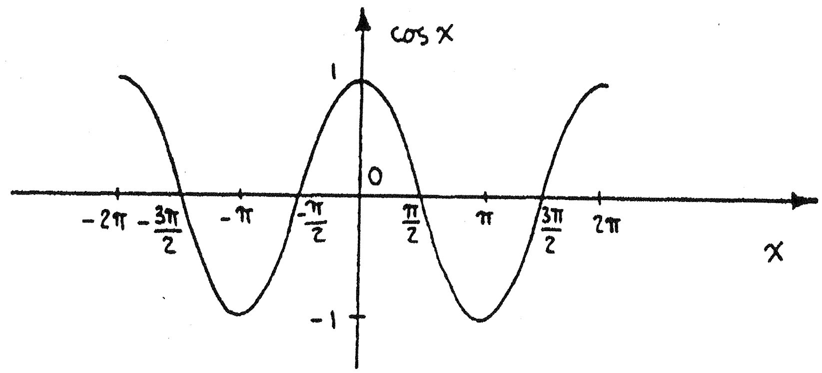
$$i(t) = -2 + 7e^{-500t}, \quad t \geq 0$$

For all t,

$$i(t) = \begin{cases} 5 & \text{for } t < 0 \\ -2+7e^{-500t} & \text{for } t \geq 0 \end{cases}$$

The Sinusoidal Source

Consider the graph of $\cos x$ where x is an angle in radians.



By letting $x = \omega t$, the angle becomes a function of time, where ω is termed the radian or angular frequency in radians per second.

The time to complete one cycle of the sinusoid is termed the period of the sinusoid and is denoted by T. Therefore,

$$T = t = \frac{x}{\omega} = \frac{2\pi}{\omega} \text{ seconds / cycle}$$

Also,

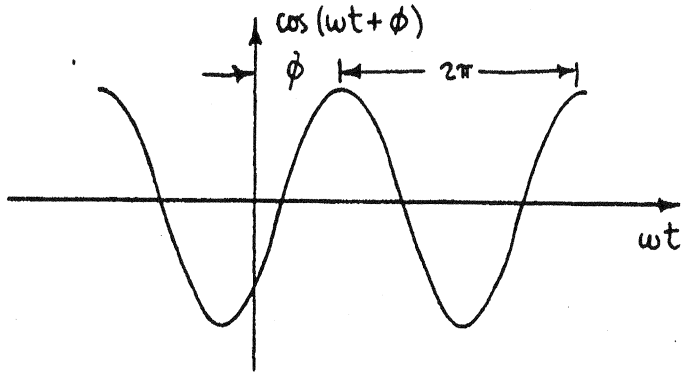
$$f = \frac{1}{T} = \frac{\omega}{2\pi} \text{ cycles / second or Hertz}$$

The relationships between real and radian frequency are

$$\boxed{f = \frac{\omega}{2\pi} \text{ Hertz}} \qquad \boxed{\omega = 2\pi f \text{ radians per second}}$$

A more general sinusoid is $\cos(\omega t + \phi)$, where ϕ is termed the phase angle of the sinusoid. When ϕ is a negative quantity, the sinusoid is translated (shifted) to the right by the amount ϕ. If ϕ is a positive quantity, the shift is to the left.

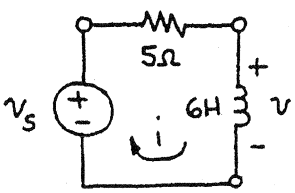
The graph of $\cos(\omega t + \phi)$ is



for negative ϕ. Also, $\cos(\omega t - \frac{\pi}{2}) = \sin \omega t$ and $\sin(\omega t + \frac{\pi}{2}) = \cos \omega t$.

Example :

Find the forced response $v(t)$ of the following circuit.



$v_s(t) = 12 \cos 2t + 7 \sin 2t$ V

Using KVL,

$$L \frac{di}{dt} + Ri = v_s$$

$$6 \frac{di}{dt} + 5i = 12 \cos 2t + 7 \sin 2t$$

$$\frac{di}{dt} + \frac{5}{6} i = 2 \cos 2t + \frac{7}{6} \sin 2t$$

the solution form for $i(t)$ must be

$$i(t) = A_1 \cos 2t + A_2 \sin 2t$$

Therefore,

$$(-2A_1 \sin 2t + 2A_2 \cos 2t) + \frac{5}{6}(A_1 \cos 2t + A_2 \sin 2t) = 2 \cos 2t + \frac{7}{6} \sin 2t$$

$$\left(\frac{5}{6} A_1 + 2A_2\right) \cos 2t + \left(\frac{5}{6} A_2 - 2A_1\right) \sin 2t = 2 \cos 2t + \frac{7}{6} \sin 2t$$

and,

$$\begin{cases} \frac{5}{6}A_1 + 2A_2 = 2 \\ -2A_1 + \frac{5}{6}A_2 = \frac{7}{6} \end{cases} \Rightarrow \begin{cases} 2A_1 + \frac{24}{5}A_2 = \frac{24}{5} \\ -2A_1 + \frac{5}{6}A_2 = \frac{7}{6} \end{cases}$$

$\therefore$

$$\frac{169}{30}A_2 = \frac{179}{30} \qquad 2A_1 = \frac{5}{6}A_2 - \frac{7}{6} = \frac{5}{6}\left(\frac{179}{169}\right) - \frac{7}{6}\left(\frac{169}{169}\right) = -\frac{48}{169}$$

$$A_2 = \frac{179}{169} \qquad A_1 = -\frac{24}{169}$$

Finally,

$$i(t) = A_1 \cos 2t + A_2 \sin 2t$$

$$= -\frac{24}{169}\cos 2t + \frac{179}{169}\sin 2t \text{ A}$$

and

$$v(t) = L\frac{di(t)}{dt}$$

$$= 6\left(\frac{48}{169}\sin 2t + \frac{358}{169}\cos 2t\right)$$

$$= \frac{2,148}{169}\cos 2t + \frac{288}{169}\sin 2t$$

$$= 12.71\cos 2t + 1.70\sin 2t$$

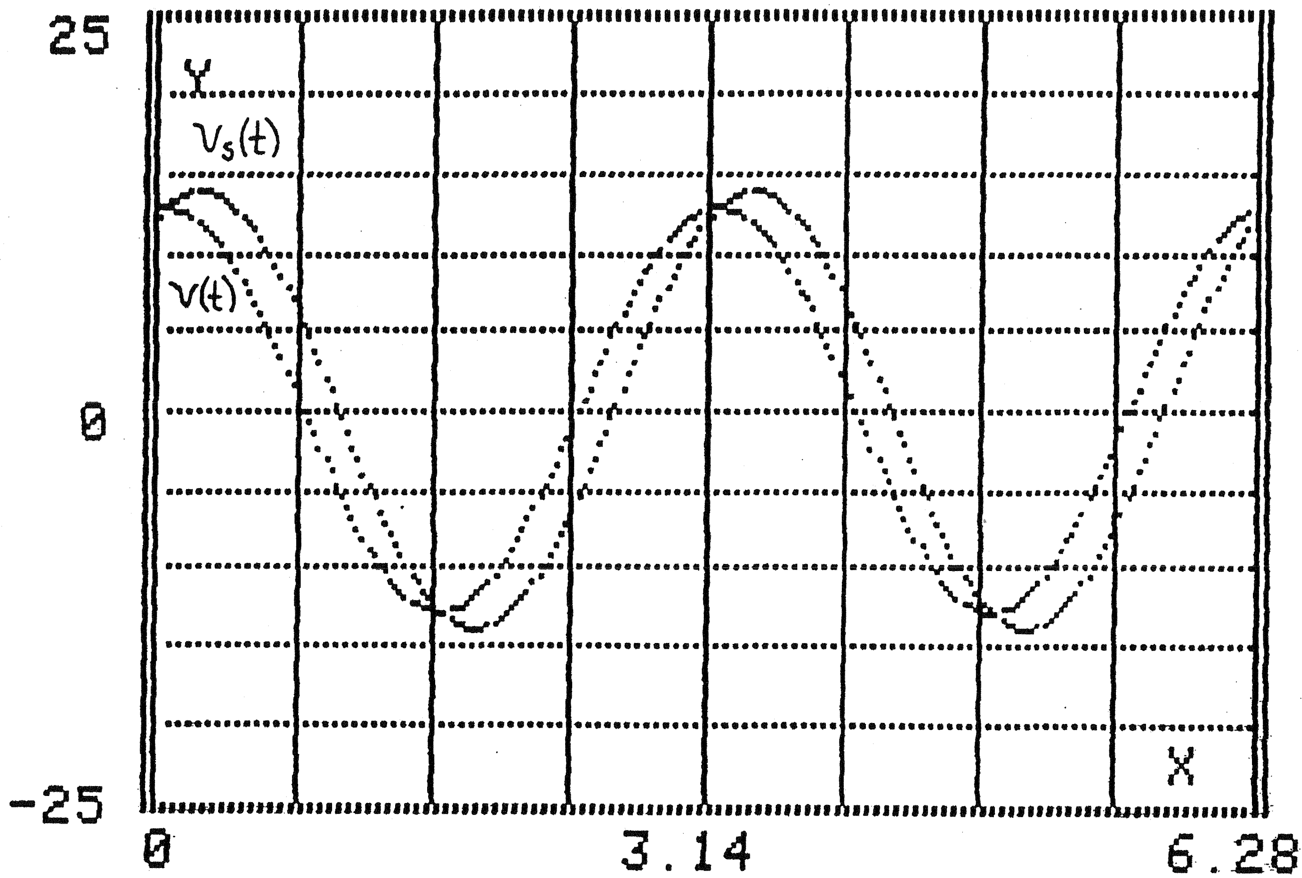
$$= \underline{\underline{12.82\cos(2t - 7.62°) \text{ V}}} \leftarrow$$

Rewriting $v_s(t)$,

$$v_s(t) = 12\cos 2t + 7\sin 2t$$

$$= 13.89\cos(2t - 30.26°)$$

and $v(t)$ <u>leads</u> $v_s(t)$ by $22.64°$.



$$v(t) = 12.82 \cos\left(2t - 7.62^\circ\right)\ \mathrm{V}$$

$$v_s(t) = 13.89 \cos\left(2t - 30.26^\circ\right)\mathrm{V}$$

In the previous example problem, a sinusoid that was represented in quadrature form as $12.71 \cos 2t + 1.70 \sin 2t$ was converted to the standard form $12.82 \cos(2t - 7.62°)$. To understand this conversion, consider the following trigonometric identity.

$$A \cos(\omega t - \theta) = A \cos \omega t \cos \theta + A \sin \omega t \sin \theta$$

Then,

$$A \cos(\omega t - \theta) = (A \cos \theta) \cos \omega t + (A \sin \theta) \sin \omega t$$

$$= x \cos \omega t + y \sin \omega t$$

Now,

$$\frac{A \sin \theta}{A \cos \theta} = \tan \theta = \frac{y}{x}$$

$$\boxed{\theta = \tan^{-1} \frac{y}{x}}$$

And,

$$A^2 \cos^2 \theta + A^2 \sin^2 \theta = x^2 + y^2$$

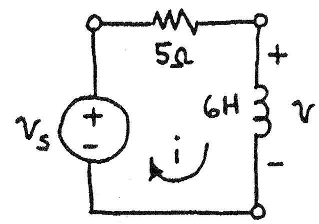
$$A^2 (\cos^2 \theta + \sin^2 \theta) = x^2 + y^2$$

$$\boxed{A = \sqrt{x^2 + y^2}}$$

The Phasor Transform

Phasor transforms permit sinusoidal circuits to be analyzed in the frequency domain in a manner analogous to resistive circuits by using the phasor versions of KCL, KVL, nodal analysis, mesh analysis, etc.

As an example, consider the earlier problem.



$$v_s(t) = 12\cos 2t + 7\sin 2t$$
$$= 13.89\cos(2t - 30.26^\circ)$$
$$= \mathrm{Re}\left[13.89\,e^{j(2t-30.26^\circ)}\right]$$
$$= Ae^{j(\omega t+\theta)}$$

The forced response to this complex sinusoid has the form $Be^{j(\omega t+\phi)}$.
Therefore,

$$6\frac{di}{dt} + 5i = 13.89\,e^{j(2t-30.26^\circ)}$$

$$\frac{di}{dt} + \frac{5}{6}i = 2.32\,e^{j(2t-30.26^\circ)}$$

$$\frac{d}{dt}\left[Be^{j(2t+\phi)}\right] + \frac{5}{6}\left[Be^{j(2t+\phi)}\right] = 2.32\,e^{j(2t-30.26^\circ)}$$

$$j2Be^{j2t}e^{j\phi} + \frac{5}{6}Be^{j2t}e^{j\phi} = 2.32\,e^{j2t}e^{-j30.26^\circ}$$

Dividing by the common factor e^{j2t},

$$\left(j2 + \frac{5}{6}\right)Be^{j\phi} = 2.32\,e^{-j30.26^\circ}$$

$$(5 + j12)\,Be^{j\phi} = 13.89\,e^{-j30.26^\circ}$$

$$Be^{j\phi} = \frac{13.89\,e^{-j30.26^\circ}}{5 + j12}$$

Multiplying the numerator and denominator by the conjugate $5-j12$,

$$Be^{j\phi} = \frac{13.89\,(5-j12)}{169}\, e^{-j30.26^\circ}$$

$$= (0.41 - j0.99)\, e^{-j30.26^\circ}$$

$$= 1.07\, e^{-j67.50^\circ}\, e^{-j30.26^\circ}$$

$$= 1.07\, e^{-j97.76^\circ}$$

Thus,

$$i(t) = Be^{j(\omega t + \phi)} = 1.07\, e^{j(2t - 97.76^\circ)}$$

and

$$v(t) = L\,\frac{di(t)}{dt}$$

$$= 6\frac{d}{dt}\left[1.07\, e^{j(2t-97.76^\circ)}\right]$$

$$= 6\frac{d}{dt}\left[1.07 e^{j2t}\, e^{-j97.76^\circ}\right]$$

$$= j6\,(1.07)\,2e^{j2t}\, e^{-j97.76^\circ}$$

$$= j\,12.84\, e^{j(2t-97.76^\circ)}$$

$$= 12.84 e^{j90^\circ} e^{j(2t-97.76^\circ)}$$

$$= 12.84\, e^{j(2t-7.76^\circ)}$$

Finally,

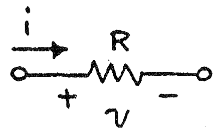
$$v(t) = \mathrm{Re}\left[12.84\, e^{j(2t-7.76^\circ)}\right]$$

$$= \underline{\underline{12.84\cos(2t - 7.76^\circ)\ \mathrm{V}}} \longleftarrow$$

A simpler notation results when the $e^{j\omega t}$ factor is suppressed.

$$Ae^{j(\omega t+\theta)} = Ae^{j\theta}e^{j\omega t} \Rightarrow Ae^{j\theta} = A\angle\theta$$

For a resistor,



and

$$v = Ri \qquad \text{(time domain)}$$

$$Ve^{j(\omega t+\phi)} = RIe^{j(\omega t+\theta)}$$

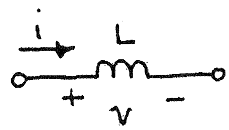
$$Ve^{j\omega t}e^{j\phi} = RIe^{j\omega t}e^{j\theta}$$

$$Ve^{j\phi} = RIe^{j\theta} \qquad \text{(frequency domain)}$$

$$V\angle\phi = RI\angle\theta$$

$$\boxed{\bar{V} = R\bar{I}} \qquad \text{(phasor equation)}$$

For an inductor,



$$v = L\frac{di}{dt}$$

$$Ve^{j(\omega t+\phi)} = L\frac{d}{dt}\left[Ie^{j(\omega t+\theta)}\right]$$

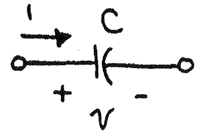
$$Ve^{j\omega t}e^{j\phi} = j\omega LIe^{j\omega t}e^{j\theta}$$

$$Ve^{j\phi} = j\omega LIe^{j\theta}$$

$$V\angle\phi = j\omega LI\angle\theta$$

$$\boxed{\bar{V} = j\omega L\bar{I}}$$

And a capacitor,



$$i = C\frac{dv}{dt}$$

$$Ie^{j(\omega t+\theta)} = C\frac{d}{dt}\left[Ve^{j(\omega t+\phi)}\right]$$

$$Ie^{j\omega t}e^{j\theta} = j\omega C\,Ve^{j\omega t}e^{j\phi}$$

$$Ie^{j\theta} = j\omega C\,Ve^{j\phi}$$

$$I\angle\theta = j\omega C\,V\angle\phi$$

$$\boxed{\bar{I} = j\omega C\,\bar{V}}$$

or

$$\boxed{\bar{V} = \frac{1}{j\omega C}\,\bar{I}}$$

In general, each phasor equation is of the form

$$\boxed{\bar{V} = Z\bar{I}}$$

where Z is the <u>impedance</u> of the element. Specifically,

$$\boxed{Z_R = R} \qquad \boxed{Z_L = j\omega L} \qquad \boxed{Z_C = \frac{1}{j\omega C}}$$

Both KCL and KVL hold for current and voltage phasors respectively. For example, in the time domain

$$i_1(t) + i_2(t) + i_3(t) + \cdots + i_n(t) = 0$$

For sinusoids,

$$I_1 \cos(\omega t + \theta_1) + I_2 \cos(\omega t + \theta_2) + \cdots + I_n \cos(\omega t + \theta_n) = 0$$

$$\mathrm{Re}\left[I_1 e^{j(\omega t + \theta_1)}\right] + \mathrm{Re}\left[I_2 e^{j(\omega t + \theta_2)}\right] + \cdots + \mathrm{Re}\left[I_n e^{j(\omega t + \theta_n)}\right] = 0$$

Dividing by $e^{j\omega t}$,

$$I_1 e^{j\theta_1} + I_2 e^{j\theta_2} + I_3 e^{j\theta_3} + \cdots + I_n e^{j\theta_n} = 0$$

$$I_1 \angle\theta_1 + I_2 \angle\theta_2 + I_3 \angle\theta_3 + \cdots + I_n \angle\theta_n = 0$$

and in the frequency domain,

$$\bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \cdots + \bar{I}_n = 0$$

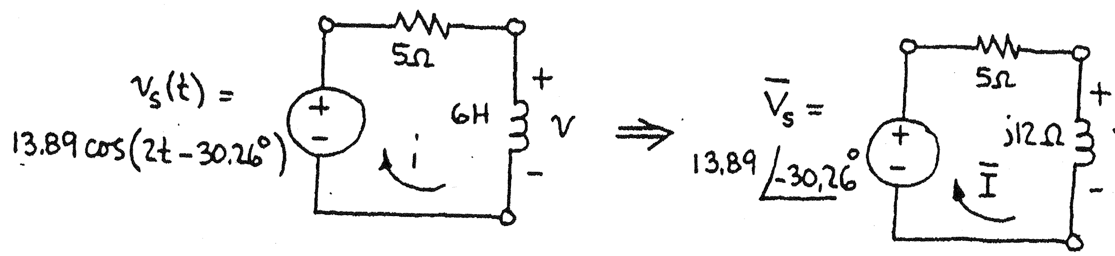
Similarly, when

$$v_1(t) + v_2(t) + v_3(t) + \cdots + v_n(t) = 0$$

it follows that

$$\bar{V}_1 + \bar{V}_2 + \bar{V}_3 + \cdots + \bar{V}_n = 0$$

Now using the phasor transform on the earlier problem,



By voltage division,

$$\bar{V} = \frac{j12}{5+j12}\bar{V}_s = \frac{12\angle 90°}{13\angle 67.38°}\,13.89\angle -30.26°$$

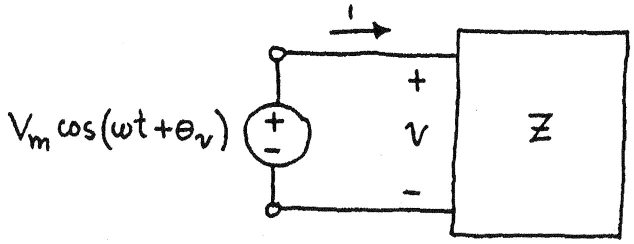
$$= \underline{\underline{12.82\angle -7.64°\ \text{V}}}$$

In the time domain,

$$v(t) = \underline{\underline{12.82\cos(2t - 7.64°)\ \text{V}}}$$

Average Power

Consider the following circuit with arbitrary impedance Z.



The current will also be a sinusoid, say $i = I_m \cos(\omega t + \theta_i)$, and the instantaneous power is

$$p = vi$$

$$= \left[V_m \cos(\omega t + \theta_v)\right]\left[I_m \cos(\omega t + \theta_i)\right]$$

By definition, the average power is

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p\,dt$$

Integration yields

$$\boxed{P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$

If $Z = R$ then $\theta_v = \theta_i$, and

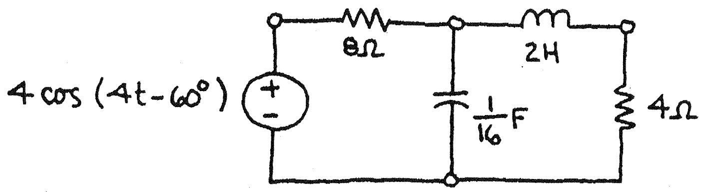
$$\boxed{P_R = \frac{1}{2} I_m^2 R} \qquad \boxed{P_R = \frac{1}{2}\frac{V_m^2}{R}}$$

If $Z = \pm jX$ then $\theta_v - \theta_i = \pm 90°$, and

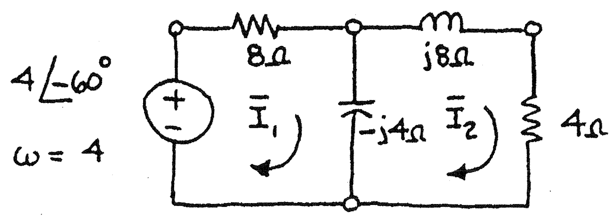
$$\boxed{P_X = 0}$$

Example:

Find the average power dissipated by each resistor.



In the frequency domain,



By inspection,

$$\bar{I}_1 = \frac{\begin{vmatrix} 4\angle{-60^\circ} & j4 \\ 0 & (4+j4) \end{vmatrix}}{\begin{vmatrix} (8-j4) & j4 \\ j4 & (4+j4) \end{vmatrix}} = \frac{\begin{vmatrix} 1\angle{-60^\circ} & j \\ 0 & (1+j) \end{vmatrix}}{\begin{vmatrix} (2-j) & j \\ j & (1+j) \end{vmatrix}} = \frac{1+j}{4+j}\, 1\angle{-60^\circ}$$

$$= \frac{(1.414\angle 45^\circ)(1\angle{-60^\circ})}{4.12\angle 14.04^\circ} = 0.34\angle{-29.04^\circ}\ \text{A}$$

$$\bar{I}_2 = \frac{\begin{vmatrix} (8-j4) & 4\angle{-60^\circ} \\ j4 & 0 \end{vmatrix}}{\begin{vmatrix} (8-j4) & j4 \\ j4 & (4+j4) \end{vmatrix}} = \frac{\begin{vmatrix} (2-j) & 1\angle{-60^\circ} \\ j & 0 \end{vmatrix}}{\begin{vmatrix} (2-j) & j \\ j & (1+j) \end{vmatrix}} = \frac{-j}{4+j}\, 1\angle{-60^\circ}$$

$$= \frac{(1\angle{-90^\circ})(1\angle{-60^\circ})}{4.12\angle 14.04^\circ} = 0.24\angle{-164.04^\circ}\ \text{A}$$

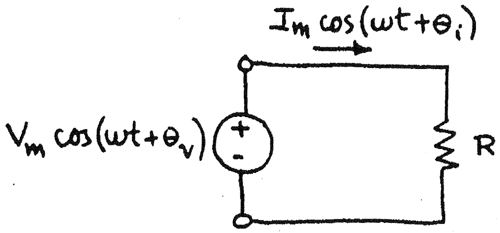
Therefore,

$$P_{8\Omega} = \frac{1}{2} I_m^2 (8) = 4|\bar{I}_1|^2 = 4(0.34)^2 = \underline{\underline{0.46\,W}}$$

$$P_{4\Omega} = \frac{1}{2} I_m^2 (4) = 2|\bar{I}_2|^2 = 2(0.24)^2 = \underline{\underline{0.12\,W}}$$

Effective Values

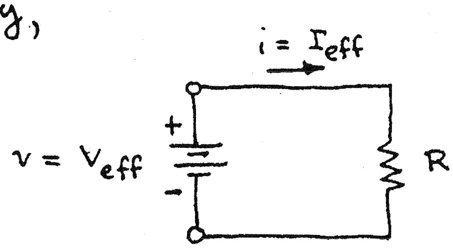
Consider the following circuit with a sinusoidal source.



The average power is

$$P_R = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} \frac{V_m^2}{R}$$

For a battery,



and the average power is

$$P_R = V_{eff} I_{eff} = I_{eff}^2 R = \frac{V_{eff}^2}{R}$$

If both sources are to produce the same power dissipation in R, there must be some effective value of I_m and V_m for the sinusoidal source that is equivalent to the dc source (battery).

$$I_{eff}^2 R = \frac{1}{2} I_m^2 R \qquad \frac{V_{eff}^2}{R} = \frac{1}{2} \frac{V_m^2}{R}$$

$$\boxed{I_{eff} = \frac{I_m}{\sqrt{2}}} \qquad \boxed{V_{eff} = \frac{V_m}{\sqrt{2}}}$$

Therefore,

$$P_R = \frac{1}{2} V_m I_m = \frac{1}{2}\left(\sqrt{2}\, V_{eff}\right)\left(\sqrt{2}\, I_{eff}\right) = V_{eff}\, I_{eff}$$

and,

$$\boxed{P_R = V_{eff}\, I_{eff} = I_{eff}^2 R = \frac{V_{eff}^2}{R}}$$

For an arbitrary impedance Z,

$$\boxed{P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{eff}\, I_{eff} \cos(\theta_v - \theta_i)}$$

For a nonsinusoidal voltage across a resistor or current through a resistor, the average power absorbed is

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p\,dt = \frac{1}{T}\int_{t_0}^{t_0+T} R i^2\,dt = R I_{eff}^2$$

Solving for I_{eff},

$$I_{eff} = \underbrace{\sqrt{}}_{\text{square \underline{root}}}\ \underbrace{\frac{1}{T}\int_{t_0}^{t_0+T}}_{\text{average or \underline{mean}}} \underbrace{i^2}_{\text{\underline{square}}}\,dt$$

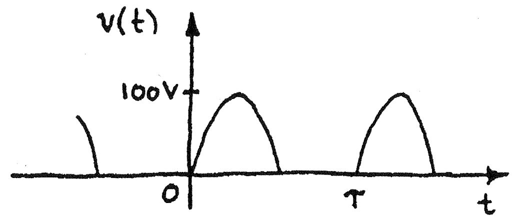
and the effective value is also known as the root-mean-square or <u>rms</u> value.

Also,

$$V_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2\,dt}$$

Example:

Determine the effective value of the following half-wave rectified sine.



$$V_{eff} = \sqrt{\frac{1}{T}\int_0^{T/2} v^2\,dt} = \sqrt{\frac{1}{T}\int_0^{T/2}\left[100\sin\left(\frac{2\pi}{T}t\right)\right]^2 dt}$$

$$= 100\sqrt{\frac{1}{T}\int_0^{T/2}\frac{1}{2}\left[1-\cos\left(\frac{4\pi}{T}t\right)\right]dt}$$

$$= 100\sqrt{\frac{1}{T}\left(\frac{1}{2}t\right)\Big|_0^{T/2} - \frac{1}{T}\left(\frac{1}{2}\right)\left(\frac{T}{4\pi}\right)\sin\left(\frac{4\pi}{T}t\right)\Big|_0^{T/2}}$$

$$= 100\sqrt{\frac{1}{4} - 0 - \frac{1}{8\pi}\left(\sin 2\pi - \sin 0\right)} = \frac{100}{2} = \underline{\underline{50V}} \leftarrow$$

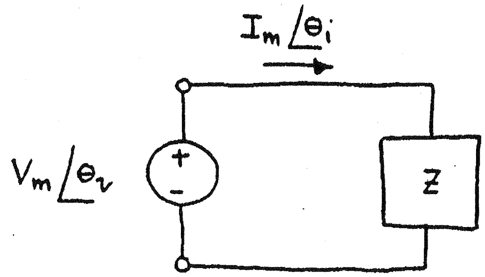
The average or dc value is

$$V_{dc} = \frac{1}{T}\int_0^{T/2} v\,dt = \frac{1}{T}\int_0^{T/2} 100\sin\left(\frac{2\pi}{T}t\right)dt = -\frac{100}{T}\left(\frac{T}{2\pi}\right)\cos\left(\frac{2\pi}{T}t\right)\Big|_0^{T/2}$$

$$= -\frac{50}{\pi}\left(\cos\pi - \cos 0\right) = -\frac{50}{\pi}(-1-1) = \frac{100}{\pi} = \underline{\underline{31.83V}} \leftarrow$$

Power Factor

Once again, consider the following circuit with arbitrary impedance Z.



$$Z = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle \theta$$

The average power absorbed by the load impedance is

$$P = \frac{1}{2} V_m I_m \cos\theta = V_{eff} I_{eff} \cos\theta = (\text{apparent power})(\cos\theta)$$

where

$$\cos\theta = \frac{P}{V_{eff} I_{eff}} = \frac{\text{average power}}{\text{apparent power}} = \underline{\text{power factor}}\ (\text{pf})$$

The angle $\theta = \theta_v - \theta_i$ is called the <u>power factor angle</u>.

If the load impedance is <u>inductive</u>, $X > 0$ and

$$\theta = \text{ang } Z = \tan^{-1} \frac{X}{R} > 0$$

The current <u>lags</u> the voltage resulting in a <u>lagging power factor</u>. If the load impedance is <u>capacitive</u>, $X < 0$ and

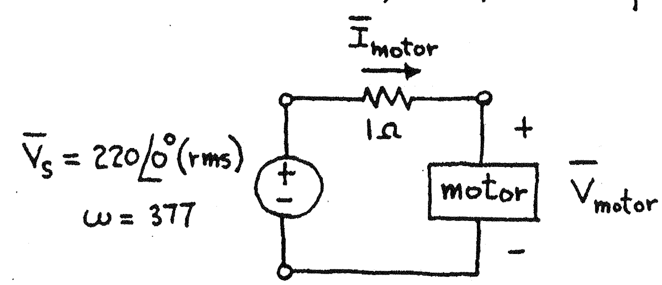
$$\theta = \text{ang } Z = \tan^{-1} \frac{X}{R} < 0$$

Now the current <u>leads</u> the voltage producing a <u>leading power factor</u>.

In each case, the current is referenced to the voltage to determine whether the power factor is lagging or leading.

Example:

A 1,000 W electric motor is designed to run from 220 V (rms) @ 60 Hz and has a lagging power factor of 0.8. If the motor is connected to the source through a 1Ω resistor, determine (a) the "line loss" and (b) the percent reduction in power loss with a 50μF capacitor in parallel with the motor.



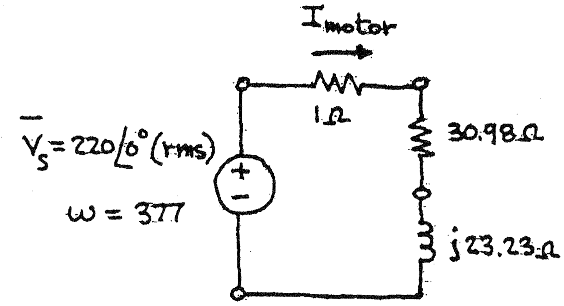
The equivalent circuit of the motor is

$$Z_{motor} = \frac{\bar{V}_{motor}}{\bar{I}_{motor}} = \frac{V_{eff}}{I_{eff}}\angle \cos^{-1} 0.8 = \frac{V_{eff}^2\,(pf)}{P}\angle 36.87° = \frac{(220)^2\,0.8}{1,000}\angle 36.87°$$

$$= 38.72\angle 36.87° = 30.98 + j\,23.23\,\Omega$$

The motor represents an inductive load.

(a)



solving for $\bar{I}_{motor}$,

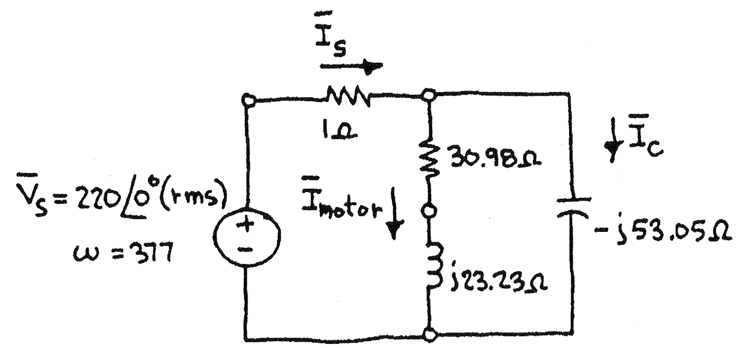
$$\bar{I}_{motor} = \frac{\bar{V}_s}{Z_{motor}+1} = \frac{220\angle 0°}{31.98 + j\,23.23} = \frac{220\angle 0°}{39.53\angle 35.99°} = 5.57\angle -35.99°\ \text{(rms) A}$$

The line loss is

$$P_L = I_{eff}^2\,R = (5.57)^2\,1 = \underline{\underline{31.02\ W}} \leftarrow$$

with the 50μF capacitor connected,

(b)



The new equivalent load is

$$Z_{eq} = \frac{(38.72\angle 36.87^\circ)(53.05\angle -90^\circ)}{30.98 - j29.82} = \frac{2{,}054.10\angle -53.13^\circ}{43.00\angle -43.91^\circ}$$

$$= 47.77\angle -9.22^\circ = 47.15 - j7.65\ \Omega$$

which is capacitive. the new current drawn from the source is

$$\bar{I}_s = \frac{\bar{V}_s}{Z_{eq}+1} = \frac{220\angle 0^\circ}{48.15 - j7.65} = \frac{220\angle 0^\circ}{48.75\angle -9.03^\circ} = 4.51\angle 9.03^\circ \text{ (rms) A}$$

The power factor for the new load (the motor in parallel with the capacitor) is

$$pf = \cos(-9.22^\circ) = 0.99 \text{ leading}$$

the new line loss is

$$P_L = I_{eff}^2 R = (4.51)^2 1 = \underline{\underline{20.34\text{ W}}} \leftarrow$$

the percent reduction in power loss is

$$\Delta\%\ P_L = \frac{31.02 - 20.34}{31.02} \times 100\% = \underline{\underline{34.43\%}} \leftarrow$$

Complex Power

Average or real power can be generalized with the notion of complex power. Beginning with the following definition of average power,

$$P = V_{eff} I_{eff} \cos\theta = V_{eff} I_{eff} \operatorname{Re}\left[e^{j\theta}\right]$$

$$= V_{eff} I_{eff} \operatorname{Re}\left[e^{j(\theta_v - \theta_i)}\right] = V_{eff} I_{eff} \operatorname{Re}\left[e^{j\theta_v} e^{-j\theta_i}\right]$$

$$= \operatorname{Re}\left[V_{eff} I_{eff} e^{j\theta_v} e^{-j\theta_i}\right] = \operatorname{Re}\left[(V_{eff} e^{j\theta_v})(I_{eff} e^{-j\theta_i})\right]$$

$$= \operatorname{Re}\left[(V_{eff} e^{j\theta_v})(I_{eff} e^{j\theta_i})^*\right] = \operatorname{Re}\left[\bar{V}_{eff} \bar{I}_{eff}^*\right]$$

Therefore, the average power absorbed is the real part of the complex power absorbed by the load, where complex power is defined as

$$\boxed{S = \bar{V}_{eff} \bar{I}_{eff}^*}$$

To distinguish complex power from either real or reactive power, the term "<u>volt amperes</u>" is used.

Rewriting complex power,

$$S = \bar{V}_{eff} \bar{I}_{eff}^* = (V_{eff} e^{j\theta_v})(I_{eff} e^{-j\theta_i})$$

$$= V_{eff} I_{eff} e^{j(\theta_v - \theta_i)} = V_{eff} I_{eff} e^{j\theta}$$

Using Euler's identity,

$$S = V_{eff} I_{eff} (\cos\theta + j \sin\theta)$$

$$= V_{eff} I_{eff} \cos\theta + j V_{eff} I_{eff} \sin\theta$$

$$= (\text{real power}) + j\,(\text{reactive power}) = P + jQ$$

The term "<u>var</u>" is used for reactive power. It stands for <u>volt-ampere reactive</u>.

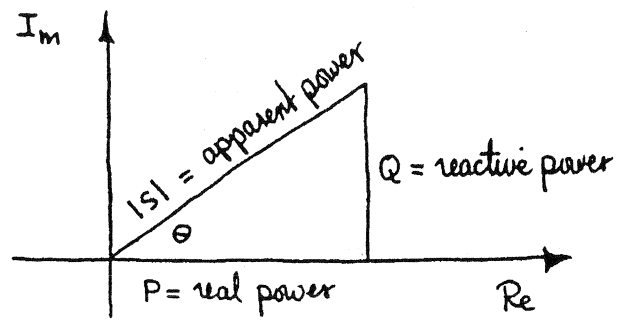
The magnitude of complex power is

$$|S| = \sqrt{P^2 + Q^2}$$

$$= \sqrt{(\text{real power})^2 + (\text{reactive power})^2}$$

$$= \text{apparent power}$$

Expressed as a (power) triangle,

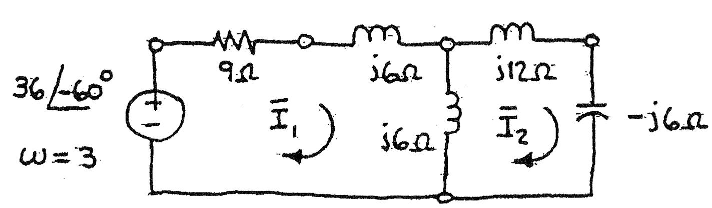


where

$$pf = \cos\theta = \frac{P}{|S|} = \frac{\text{real power}}{\text{apparent power}}$$

Example:

Consider the following circuit where $\bar{I}_1 = 2\sqrt{2}\angle{-105^\circ}$ A and $\bar{I}_2 = \sqrt{2}\angle{-105^\circ}$ A.



Determine:

(a) The complex power absorbed by the capacitor.
(b) The real power absorbed by the capacitor.
(c) The complex power absorbed by the resistor.
(d) The real power absorbed by the resistor.

(a)

$$\bar{V}_c = -j6\bar{I}_2 = (6\angle{-90^\circ})(\sqrt{2}\angle{-105^\circ}) = 6\sqrt{2}\angle{-195^\circ} = 6\sqrt{2}\angle{165^\circ}$$

$$S_c = \tfrac{1}{2}\bar{V}_c\,\bar{I}_2^* = \tfrac{1}{2}(6\sqrt{2}\angle{165^\circ})(\sqrt{2}\angle{105^\circ}) = 6\angle{270^\circ} = \underline{\underline{6\angle{-90^\circ}\text{ VA}}} \leftarrow$$

(b)

$$P_c = V_{eff}\,I_{eff}\cos\theta = V_{eff}\,I_{eff}\cos(-90^\circ) = \underline{\underline{0\text{W}}} \leftarrow$$

(c)

$$\bar{V}_R = 9\bar{I}_1 = 9(2\sqrt{2}\angle{-105^\circ}) = 18\sqrt{2}\angle{-105^\circ}$$

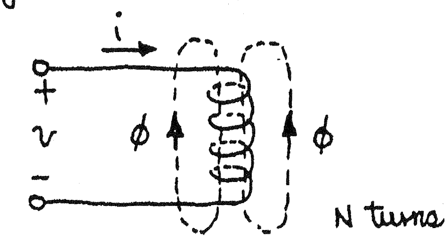
$$S_R = \tfrac{1}{2}\bar{V}_R\,\bar{I}_1^* = \tfrac{1}{2}(18\sqrt{2}\angle{-105^\circ})(2\sqrt{2}\angle{105^\circ}) = \underline{\underline{36\angle{0^\circ}\text{ VA}}} \leftarrow$$

(d)

$$P_R = \tfrac{1}{2}|\bar{I}_1|^2 R = \tfrac{1}{2}(2\sqrt{2})^2 9 = \underline{\underline{36\text{W}}} \leftarrow$$

Self-Inductance

Consider the following N-turn coil.



The right-hand rule determines the orientation of the magnetic field related to the direction of the current. Induced voltage can be expressed by Faraday's law,

$$\boxed{v = \frac{d\lambda}{dt}}$$

where λ is the flux linkage in weber-turns. Flux linkage is the product of the magnetic field ϕ in webers and the number of turns linked by the field N.

$$\boxed{\lambda = N\phi}$$

The magnitude of the flux ϕ can be written as

$$\boxed{\phi = \mathcal{P} N i}$$

where $\mathcal{P}$ is the permeance of the space occupied by the field. Combining these relationships,

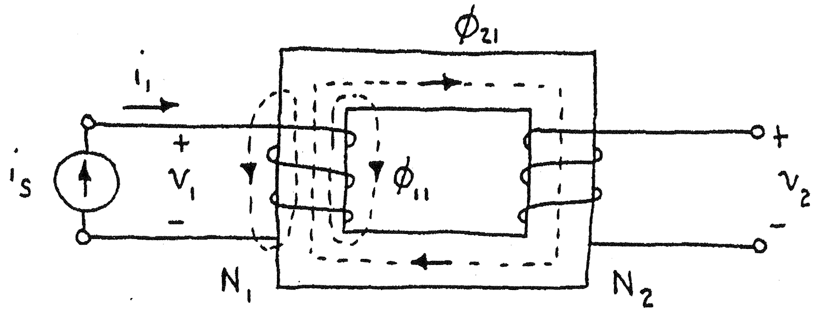
$$v = \frac{d\lambda}{dt} = \frac{d}{dt}(N\phi) = N\frac{d\phi}{dt} = N\frac{d}{dt}(\mathcal{P}Ni) = N^2\mathcal{P}\frac{di}{dt} = L\frac{di}{dt}$$

Thus self-inductance is proportional to the square of the number of turns of the coil.

$$\boxed{L = N^2\mathcal{P}}$$

Mutual Inductance

Mutual inductance is the circuit parameter that relates the voltage induced in one coil to the time-varying current in another coil. Consider the following two coils that are magnetically coupled.



Letting ϕ_1 represent the total flux linking coil 1,

$$\phi_1 = \phi_{11} + \phi_{21}$$

Also,

$$\phi_1 = \mathcal{P}_1 N_1 i_1$$

$$\phi_{11} = \mathcal{P}_{11} N_1 i_1$$

$$\phi_{21} = \mathcal{P}_{21} N_1 i_1$$

Substituting,

$$\mathcal{P}_1 = \mathcal{P}_{11} + \mathcal{P}_{21}$$

Using Faraday's law,

$$v_1 = \frac{d\lambda_1}{dt} = \frac{d}{dt}(N_1\phi_1) = N_1 \frac{d}{dt}(\phi_{11} + \phi_{21})$$

$$= N_1^2(\mathcal{P}_{11} + \mathcal{P}_{21})\frac{di_1}{dt} = N_1^2 \mathcal{P}_1 \frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

where L_1 is the self-inductance of coil 1.

$$v_2 = \frac{d\lambda_2}{dt} = \frac{d}{dt}(N_2\phi_{21}) = N_2\frac{d}{dt}(\mathcal{P}_{21} N_1 i_1)$$

$$= N_2 N_1 \mathcal{P}_{21} \frac{di_1}{dt} = M_{21} \frac{di_1}{dt}$$

The mutual inductance M_{21} relates the voltage induced in coil 2 to the current in coil 1.

$$\boxed{M_{21} = N_2 N_1 \mathcal{P}_{21}}$$

If coil 2 is excited and coil 1 is left open, a similar procedure yields

$$\boxed{M_{12} = N_1 N_2 \mathcal{P}_{12}}$$

For nonmagnetic materials, $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$ are equal. Therefore,

$$\boxed{M_{12} = M_{21} = M}$$

Mutual inductance is also a function of the self-inductances. Beginning with

$$L_1 = N_1^2 \mathcal{P}_1$$

$$L_2 = N_2^2 \mathcal{P}_2$$

it follows that

$$L_1 L_2 = N_1^2 N_2^2 \mathcal{P}_1 \mathcal{P}_2 = N_1^2 N_2^2 (\mathcal{P}_{11} + \mathcal{P}_{21})(\mathcal{P}_{22} + \mathcal{P}_{12})$$

$$= N_1^2 N_2^2 (\mathcal{P}_{11}\mathcal{P}_{22} + \mathcal{P}_{11}\mathcal{P}_{12} + \mathcal{P}_{21}\mathcal{P}_{22} + \mathcal{P}_{21}\mathcal{P}_{12})$$

$$= (N_1 N_2 \mathcal{P}_{12})^2 \left(1 + \frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right)\left(1 + \frac{\mathcal{P}_{22}}{\mathcal{P}_{12}}\right)$$

assuming $\mathcal{P}_{21} = \mathcal{P}_{12}$.

Making the following substitution,

$$\frac{1}{k^2} = \left(1 + \frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right)\left(1 + \frac{\mathcal{P}_{22}}{\mathcal{P}_{12}}\right)$$

results in

$$M^2 = k^2 L_1 L_2$$

$$\boxed{M = k\sqrt{L_1 L_2}}$$

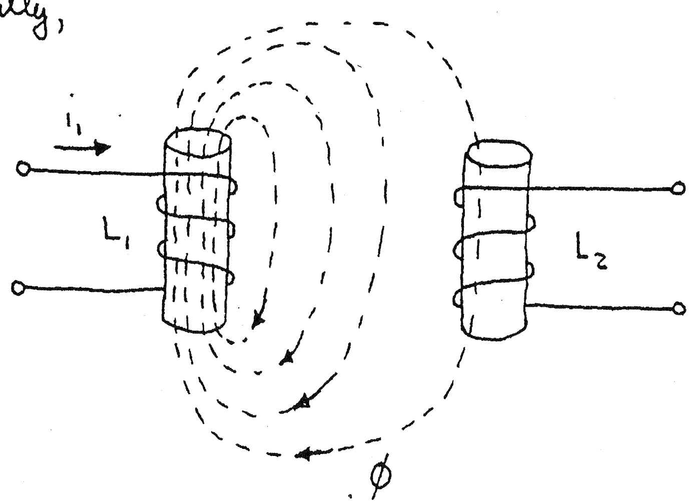
where the constant k is termed the coefficient of coupling. The limits of k are

$$\boxed{0 \leq k \leq 1}$$

The coefficient of coupling can also be expressed in the following manner.

$$\boxed{k = \frac{\text{flux linkages between } L_1 \text{ and } L_2}{\text{flux produced by } L_1}}$$

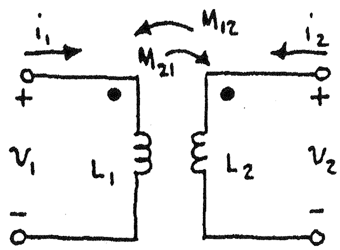
Pictorially,



which represents $k = \frac{1}{4} = 0.25$.

Polarity of Mutually Induced Voltages

Consider two coils with self-inductances L_1 and L_2 that are magnetically coupled.



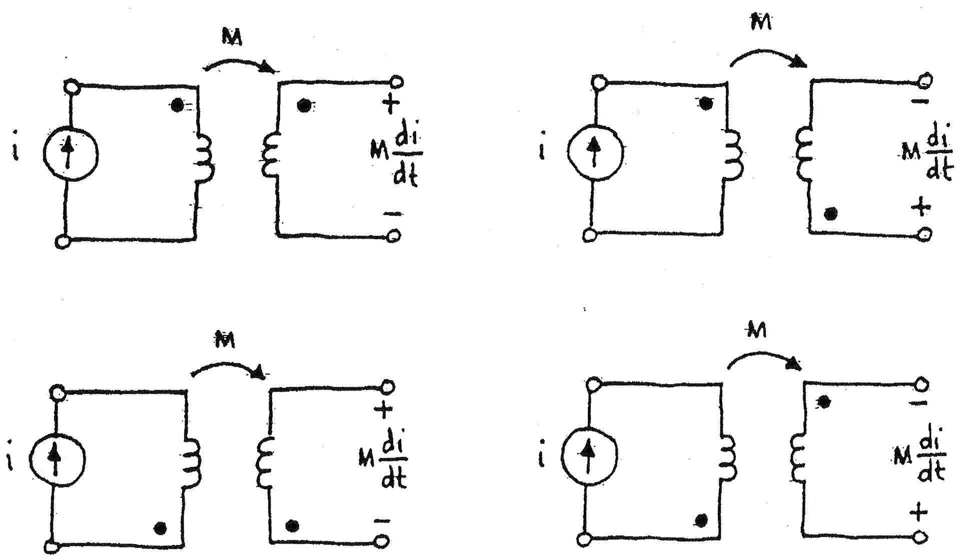
Furthermore,

$$v_2 = M_{21}\frac{di_1}{dt}, \quad i_1 \neq 0 \text{ and } i_2 = 0$$

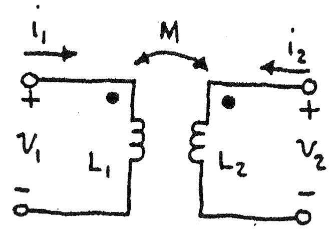
$$v_1 = M_{12}\frac{di_2}{dt}, \quad i_2 \neq 0 \text{ and } i_1 = 0$$

where M_{12} and M_{21} are the coefficients of mutual inductance or mutual inductance, for short.

The dots describe the physical orientation of the coils and indicate the phase relationship of the current in one inductor and the resulting induced voltage in the other inductor. Current flowing <u>into</u> one dot of an inductor results in a <u>positive</u> induced voltage at the dot of the other inductor.



Such an arrangement of coils produces a four-terminal device called a transformer. One side of the transformer is often called the primary winding and the other side is the secondary winding. If $M_{12} = M_{21} = M$, the symbol becomes



The equations that describe this circuit element in the time domain are

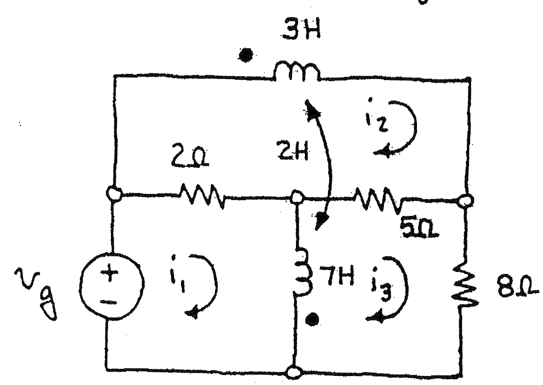
$$\boxed{\begin{aligned} v_1 &= L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} \\ v_2 &= M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \end{aligned}}$$

In the phasor domain,

$$\boxed{\begin{aligned} \bar{V}_1 &= j\omega L_1 \bar{I}_1 + j\omega M \bar{I}_2 \\ \bar{V}_2 &= j\omega M \bar{I}_1 + j\omega L_2 \bar{I}_2 \end{aligned}}$$

Example:

Write a set of mesh-current equations for the following circuit.



Around loop 1,

$$v_g = 2(i_1 - i_2) + 7\frac{d}{dt}(i_1 - i_3) - 2\frac{di_2}{dt}$$

Loop 2,

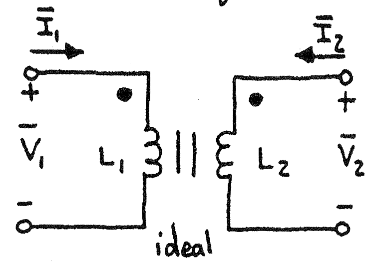
$$0 = 3\frac{di_2}{dt} + 5(i_2 - i_3) + 2(i_2 - i_1) + 2\frac{d}{dt}(i_3 - i_1)$$

Loop 3,

$$0 = 5(i_3 - i_2) + 8i_g + 7\frac{d}{dt}(i_3 - i_1) + 2\frac{di_2}{dt}$$

The Ideal Transformer

Consider the following ideal transformer in the phasor domain.



Now,

$$\bar{V}_1 = j\omega L_1 \bar{I}_1 + j\omega M \bar{I}_2$$

$$\bar{I}_1 = \frac{\bar{V}_1 - j\omega M \bar{I}_2}{j\omega L_1}$$

Also,

$$\bar{V}_2 = j\omega M \bar{I}_1 + j\omega L_2 \bar{I}_2 = j\omega M \left(\frac{\bar{V}_1 - j\omega M \bar{I}_2}{j\omega L_1} \right) + j\omega L_2 \bar{I}_2$$

$$= \frac{M \bar{V}_1}{L_1} - \frac{j\omega M^2 \bar{I}_2}{L_1} + j\omega L_2 \bar{I}_2$$

For perfect coupling,

$$K = \frac{M}{\sqrt{L_1 L_2}} = 1 \Rightarrow M^2 = L_1 L_2$$

Therefore,

$$\bar{V}_2 = \frac{\sqrt{L_1 L_2}\ \bar{V}_1}{L_1} - \frac{j\omega L_1 L_2 \bar{I}_2}{L_1} + j\omega L_2 \bar{I}_2 = \sqrt{\frac{L_2}{L_1}}\ \bar{V}_1$$

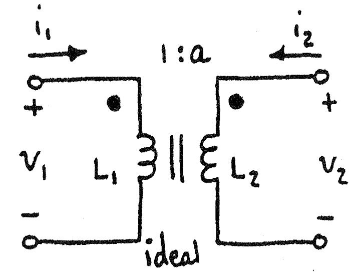
and,

$$\frac{\bar{V}_2}{\bar{V}_1} = \sqrt{\frac{L_2}{L_1}} = \sqrt{\frac{N_2^2 \cancel{\wp}}{N_1^2 \cancel{\wp}}} = \frac{N_2}{N_1} = a = \underline{\text{turns ratio}}$$

Solving for the current ratio produces

$$\frac{\bar{I}_1}{\bar{I}_2} = -\sqrt{\frac{L_2}{L_1}} = -\frac{N_2}{N_1} = -a$$

In general, for an ideal transformer,



the equations that describe this circuit element in the time domain are

$$\boxed{\begin{array}{lcl} v_2 = a v_1 & \Longleftrightarrow & v_1 = \dfrac{v_2}{a} \\ i_2 = -\dfrac{i_1}{a} & \Longleftrightarrow & i_1 = -a i_2 \end{array}}$$

In the phasor domain,

$$\boxed{\begin{array}{lcl} \bar{V}_2 = a\bar{V}_1 & \Longleftrightarrow & \bar{V}_1 = \dfrac{\bar{V}_2}{a} \\ \bar{I}_2 = -\dfrac{\bar{I}_1}{a} & \Longleftrightarrow & \bar{I}_1 = -a\bar{I}_2 \end{array}}$$

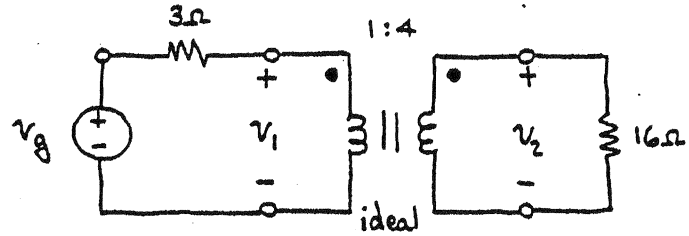
The instantaneous energy stored in an ideal transformer is

$$\begin{aligned} w(t) &= \tfrac{1}{2}L_1 i_1^2 + \tfrac{1}{2}L_2 i_2^2 + M i_1 i_2 \\ &= \tfrac{1}{2}L_1(-a i_2)^2 + \tfrac{1}{2}L_2 i_2^2 + M(-a i_2)\, i_2 \\ &= \left(\tfrac{1}{2}L_1 a^2 + \tfrac{1}{2}L_2 - Ma\right) i_2^2 \\ &= \left[\tfrac{1}{2}L_1\left(\tfrac{L_2}{L_1}\right) + \tfrac{1}{2}L_2 - \sqrt{L_1 L_2}\cdot\sqrt{\tfrac{L_2}{L_1}}\,\right] i_2^2 \\ &= \left(\tfrac{1}{2}L_2 + \tfrac{1}{2}L_2 - L_2\right) i_2^2 = 0\,\text{J} \end{aligned}$$

Therefore, the instantaneous and average powers are also zero.

Example:

The following transformer is ideal.



Determine:

(a) The impedance seen by the voltage source,

(b) The voltage gain $\bar{V}_2 / \bar{V}_g$,

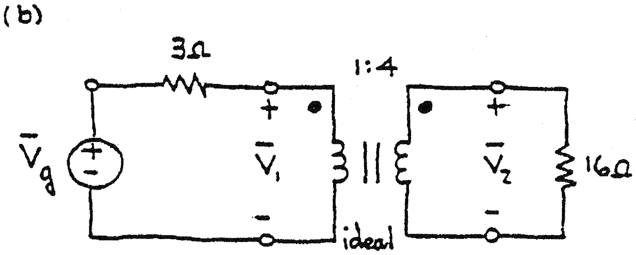
(c) The new load that will absorb maximum power, and

(d) The value of a that will result in the 16Ω resistor absorbing maximum power.

(a)

$$Z_{in} = R_1 + Z_r = R_1 + R_L\left(\frac{N_1}{N_2}\right)^2 = 3 + 16\left(\frac{1}{4}\right)^2 = \underline{\underline{4\Omega}} \leftarrow$$

(b)

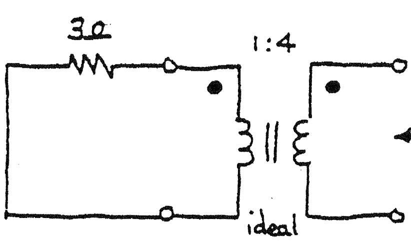


$$\bar{V}_1 = \frac{1}{1+3}\bar{V}_g = \frac{1}{4}\bar{V}_g$$

$$\bar{V}_2 = a\bar{V}_1 = 4\left(\frac{1}{4}\bar{V}_g\right) = \bar{V}_g$$

$$\therefore \quad \frac{\bar{V}_2}{\bar{V}_g} = \underline{\underline{1}} \leftarrow$$

(c)



$$\leftarrow Z_o = R_1\left(\frac{N_2}{N_1}\right)^2 = 3\left(\frac{4}{1}\right)^2 = \underline{\underline{48\Omega}} \leftarrow$$

(d)

$$Z_o = 3a^2 = R_L = 16 \Rightarrow a^2 = \frac{16}{3} \Rightarrow a = \frac{4}{\sqrt{3}} = \underline{\underline{2.31}} \leftarrow$$
