Chapter 7 Discrete time signal Description

*comparison with continuous description

7.1 discrete time signals

Come from:

<1> measurable discrete quantities; example: population

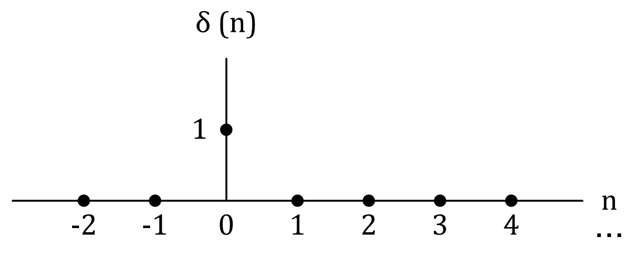
<2> sampled continuous quantities; example: electrical quantities

7.2 Typical discrete signal sequences*

a) Unit impulse (Kroneck Delta*) sequences

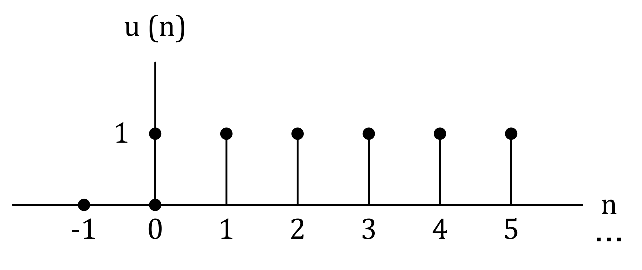
$$\delta(k-k_0)=\begin{cases}1 & k=k_0\\ 0 & k\neq k_0\end{cases}$$

Note: not a singular function!



b) Unit step sequence

$$u(k-k_0)=\begin{cases}1 & k>k_0\\ 0 & k<k_0\\ \begin{cases}1\\ \frac{1}{2}\\ 0\end{cases} & k=k_0\end{cases}$$



Relation with unit impulse*

$$\delta(k)=u(k)-u(k-1)$$

$$u(k)=\sum_{n=-\infty}^{k}\delta(n)$$

c) Unit ramp sequence

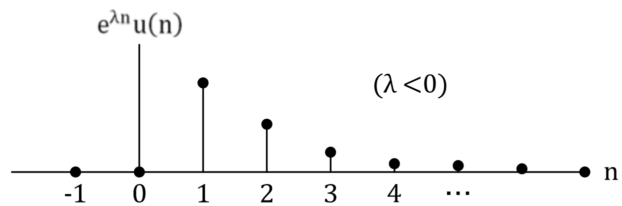
$$r(k) = \begin{cases} k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

d) Unit alternating

$$\mathrm{u}_{\pm}(k) = \begin{cases} (-1)^k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

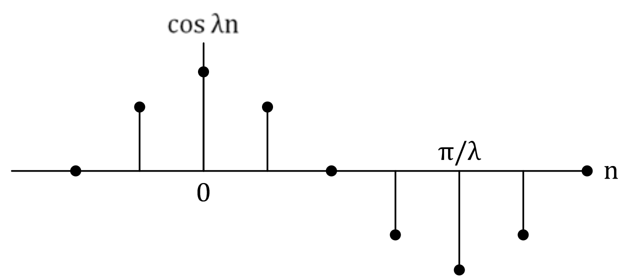
e) Unit exponential

$$e^{\lambda k} u(k)$$



f) Unit sinusoid

$$\cos \Omega k$$



g) Complex exponential

$$e^{j\Omega k} = \cos \Omega k + j \sin \Omega k$$

Is a discrete sinusoid or complex exponential periodic?
Not necessarily.

7.3 Discrete Periodic Signals

$$\underline{f(k+N) = f(k)}$$

<1> For all k

<2> Period N is the smallest number for which signal repeats!

Now look at $e^{j\Omega_0}$

$$e^{j\Omega_0(k+N)=} e^{j\Omega_0 k} e^{j\Omega_0 N} = e^{j\Omega_0 k} (if\ periodic)$$

Then $e^{j\Omega_0 N} = 1 \Rightarrow \Omega_0 N = 2\pi n$

$$\Omega_0 = \frac{n}{T} 2\pi \qquad \frac{\Omega_0}{2\pi} = \frac{n}{N}$$

The normalized frequency must be rational for the periodicity of discrete signals.

Distinct value of Ω_0 does not always produce periodic signals!

DISCRETE-TIME PERIODIC SIGNALS

A discrete-time signal $f(\text{k})$, $\text{k} = 0, \pm 1, \pm 2, \dots$ is said to be periodic with period P, where P is a positive integer, if

$$f(k) = f(k + P) \tag{7.1}$$

for all integers k in $(-\infty, \infty)$. If (7.2) holds, then

$$f(k) = f(k + P) = f(k + 2P) = \cdots = f(k + mP)$$

for any k and every positive integer m. Thus if $f(\text{k})$ is periodic with period P, it is periodic with period 2P, 3P, … The smallest such P is called the *fundamental* period. Unless stated otherwise, the period will refer to the fundamental period. The *fundamental frequency* is defined as 2π/P.

Before proceeding, we discuss some differences between sinusoidal functions and sinusoidal sequences. In the continuous-time case, $\sin \omega t$ is periodic for every ω. In the discrete-time case, however, $\sin \omega k$ may not be periodic for every ω. The condition for $\sin \omega k$ to be periodic is that there exists a positive P such that

$$\sin \omega k = \sin \omega(k + P) = \sin(\omega k + \omega P)$$

for all k. this holds if and only if

$$\omega P = 2\pi m \quad \boldsymbol{or} \quad \frac{\omega}{\pi} = \frac{2m}{P} \tag{7.2}$$

for some integer m. Thus $\sin \omega k$ is periodic if and only if ω/π is a rational number.

In other words, $\sin \omega k$ is periodic if and only if there exists an integer m such that

$$P = \frac{2m\omega}{\omega} \tag{7.3}$$

is a positive integer. The smallest such P is the fundamental period of $\sin \omega k$. For example, $\sin 2k$ is not periodic because $\frac{2}{\pi}$ is not a rational number. In this case, there exists no integer m in $P = \frac{2m\pi}{\omega}$ such that P is integer. The sequence $\sin 0.01\pi k$ is periodic because $\frac{\omega}{\pi} = \frac{0.01\pi}{\pi} = \frac{1}{100}$ is a rational number. Its period is $P = \frac{2m\pi}{\omega} = \frac{2m\pi}{0.01\pi} = 200m = 200$ by choosing $m = 1$. The sequence $\sin 3\pi k$ is periodic with period $P = \frac{2m\pi}{3\pi} = \frac{2m\pi}{3} = 2$ by choosing $m = 3$.

Consider $\sin 3.2\pi k$. It can be simplified as

$$sin\, 3.2\pi k = sin(2\pi + 1.2\pi)k = sin\, 2\pi k\, cos\, 1.2\pi k + cos\, 2\pi k\, sin\, 1.2\pi k = sin\, 1.2\pi k$$

where we have used the fact that $\sin 2\pi k = 0$ and $\cos 2\pi k = 1$ for every integer k. This implies that when we are given $\sin \omega k$, we can always reduce ω to the range $[0,2\pi)$ by subtracting or adding 2π or its multiple. Thus in the discrete-time case, we have

$$sin\, 6.2\pi k = sin\, 0.2\pi k \;\; and \;\; cos(-2.4\pi k) = cos\, 1.6\pi k$$

for all integer k. In the continuous-time case, $\sin 6.2\pi t$ and $\sin 0.2\pi t$ are two different functions.

In the continuous-time case, the fundamental frequency of $\sin \omega t$ and $\cos \omega t$ is ω in radians per second. In view of (7.4), the fundamental frequency of $\sin \omega k$ and $\cos \omega k$ may not be equal to ω. To better see the relationship between the fundamental frequency and ω, we plot in Fig. 7.1 $\cos \omega k$ and $\cos 1.9\pi k$. The sequence $\cos \pi k$ has period $P = 2$ and fundamental frequency $\frac{2\pi}{P} = \pi$. In order to find the period of $\cos 1.9\pi k$, we compute

$$P = \frac{2m\pi}{1.9\pi} = \frac{2m}{1.9}$$

The smallest integer m to make P an integer is 19. Thus the period of $\cos 1.9\pi k$ is $P = 2 \bullet \frac{19}{1.9} = 20$, and the fundamental frequency is $2\pi/20 = 0.1\pi$. We see that this fundamental frequency is smaller than the one of $\cos \pi k$, thus $\cos \pi k$ changes more rapidly than $\cos 1.9\pi k$ as shown in Fig 7.1. Thus in the discrete-time case, $\cos \omega_1 k$ may not have a higher fundamental frequency than that of $\cos \omega_2 k$ even if $\omega_1 > \omega_2$. This phenomenon does not exist in the continuous-time case. In conclusion, the fundamental frequency of $\cos \omega k$ is not necessarily equal to ω as in the continuous-time case. To compute its fundamental frequency, we must use (7.3) to compute its fundamental period P. then the fundamental frequency is equal to $2\pi/P$.

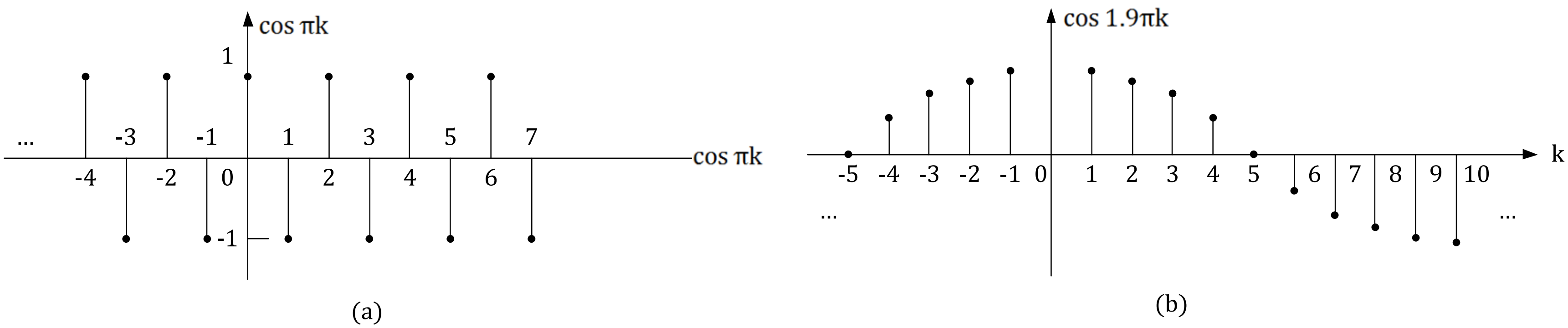


Figure 7-1

7.4 Basic operations

a) Time-reversal $y(k) = x(-k)$

b) Time-scaling $y(k) = x(ak)$

Note:

- ✧ a has to be integer!
- ✧ Time-reversal $a = -1$

c) Time-delay $y(k) = x(k - n)$

- ✧ n has to be integer!

d) Combinations $y(k) = x(m - k)$

7.5 symmetry

Even if $f(k) = f(-k)$

Odd if $f(k) = -f(-k)$

A signal does not have to be even or odd!

An arbitrary signal $f(k) = f_e(k) + f_0(k)$

$$Where\ f_e(k) = \frac{f(k) + f(-k)}{2} \quad f_O(k) = \frac{f(k) - f(-k)}{2}$$

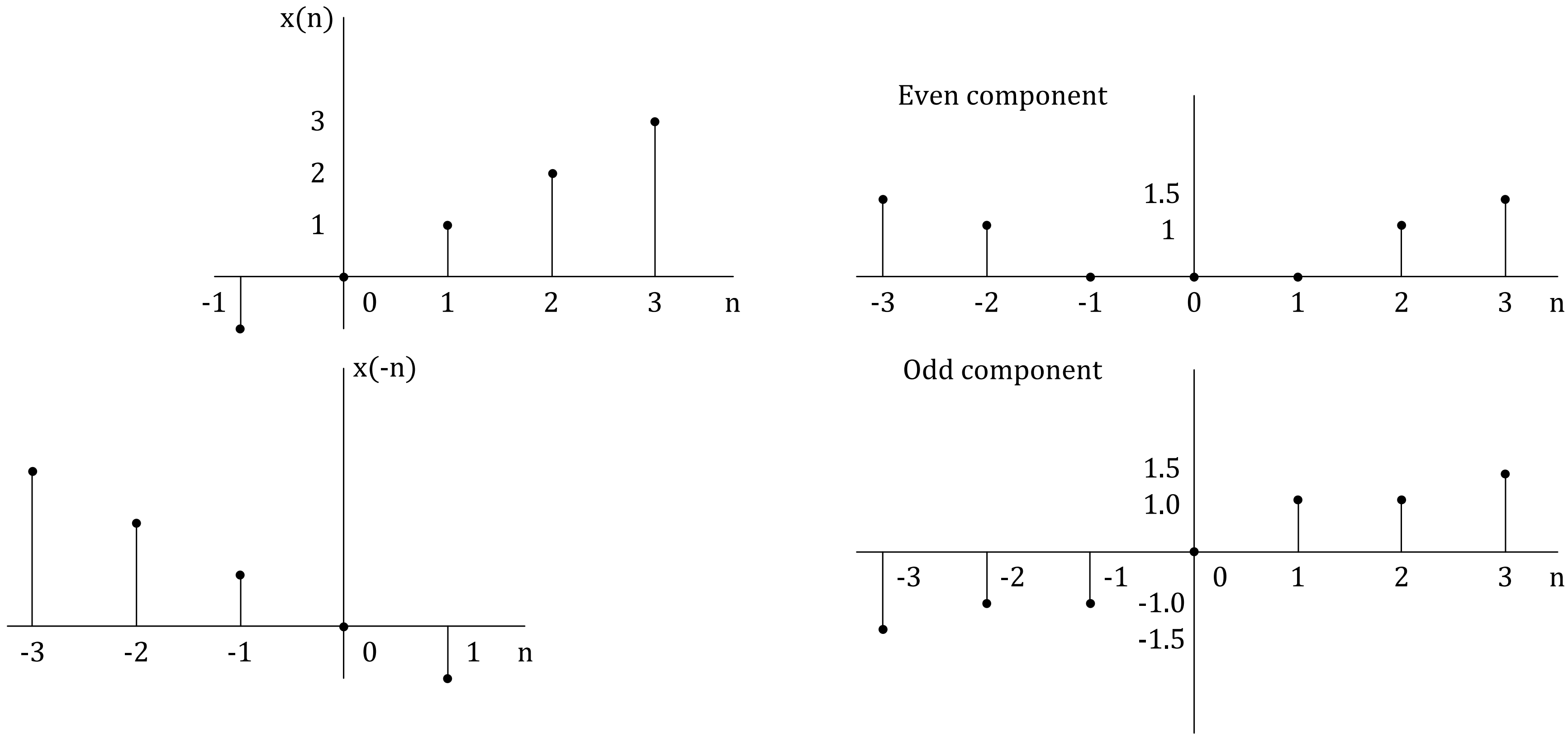


Figure 7.2 odd and even components

7.6 Representation by orthonormal functions

—Signals are often represented in terms of basis functions

—Basis functions are often chosen to be orthonormal

—Basis vectors

*Choose $\{\underline{a_1}, \underline{a_2}, \underline{a_3}\}$ to be basis vectors.

*Project rector F on basis vector, projection coefficients or weights are $\{\underline{F_1}, \underline{F_2}, \underline{F_3}\}$

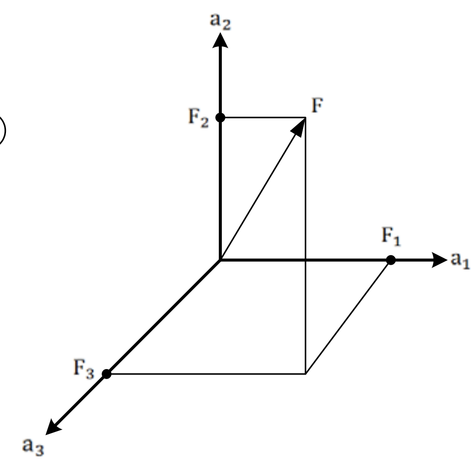
$$F = \sum_{i=1}^{3} F_i \, \underline{a_i} \quad (1)$$

orthogonality $\underline{a_i} \bullet \underline{a_j} = \begin{cases} c, & i = j \\ 0, & i \neq j \end{cases}$

normality $\underline{a_i} \bullet \underline{a_j} = 1$

Dot Product

orthonormality $\underline{a_i} \bullet \underline{a_j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$



Projection

$$F_i = F \bullet \underline{a_i} \qquad ②$$

$$If \quad \left| F - \sum_{i=1}^{3} F_i \, \underline{a_i} \right| = 0$$

Then

<i> F can be represented by $\{\underline{a_i}\}$

<ii> $\{\underline{a_i}\}$ are complete

<iii> F is in the span of $\{\underline{a_i}\}$

① and ② are similar to Fourier Series Expansion.

Basis functions $\{\varphi_i(k)\}$

$$f(k) = \sum_{i=1}^{K} c_i \, \varphi_i(k) \qquad\qquad \textit{Inner Product}$$

Condition:

$$f(k) \text{— } square\ summable \Leftrightarrow \sum_k |f(k)|, \sum_k |f(k)| \quad finite$$

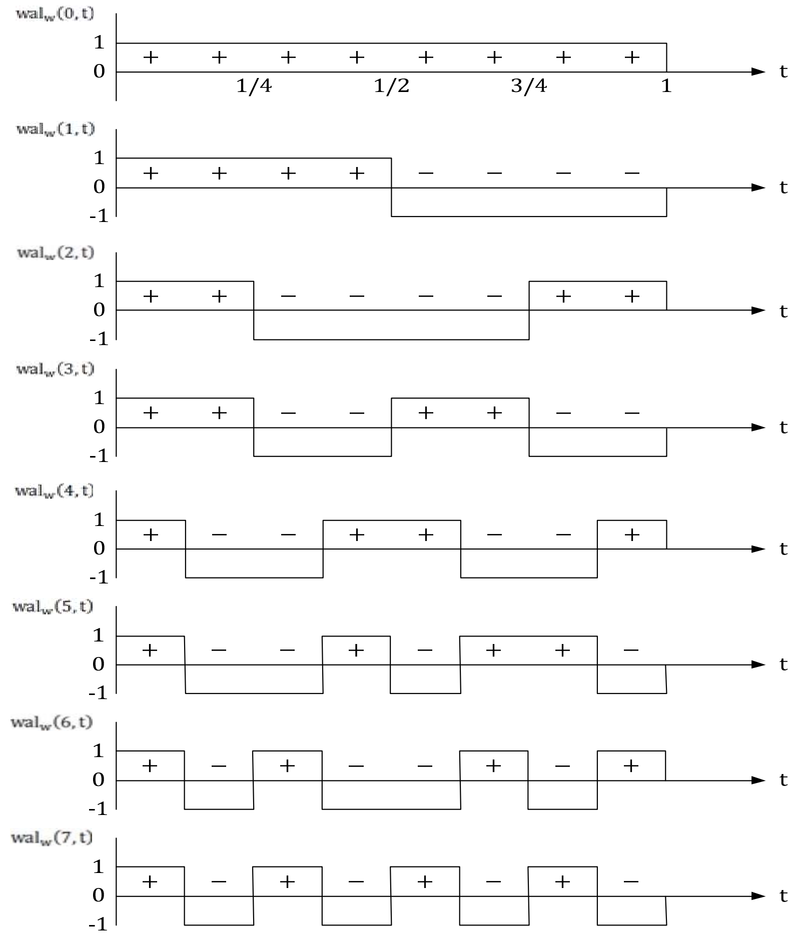
Orthonormal functions:

$$\sum_k \varphi_i(k) \, \varphi_j{}^*(k) = \begin{cases} \|\varphi_i\|^2 = 1 & i = j \\ 0 & i \neq j \end{cases}$$

Look at

$$\sum_k f(k) \, \varphi_j{}^*(k) = \sum_k \sum_i c_i \, \varphi_i(k) \varphi_j{}^*(k) = \sum_i c_i \sum_k \varphi_i(k) \varphi_j{}^*(k) = c_j$$

Walsh Function



Walsh Functions; Sequency Ordered, N = 8

Figure 7-3 Walsh Functions

Examples

— $\cos k \frac{2\pi}{N}$ $\qquad k \in (0, N-1)$

— $e^{jk\frac{2\pi}{N}}$ $\qquad k \in (0, N-1)$

— *Walsh Function*

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

— *Weighted orthonormal function*

$$\sum_k \omega(k)\, \Phi_i(k)\Phi_j(k) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Basis function can be chosen as

$$\varphi_i(k) = \sqrt{\omega(k)}\Phi_i(k)$$