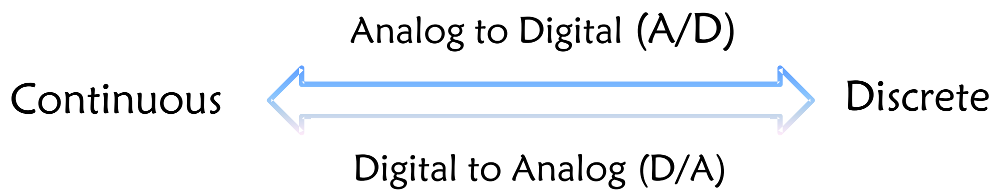
Linear Analysis of Signals, Systems, and Transform II:
Discrete signals, Systems, and Transforms

Chapter 6 Sampling of Signals

6.1 Conversion between continuous and discrete signals



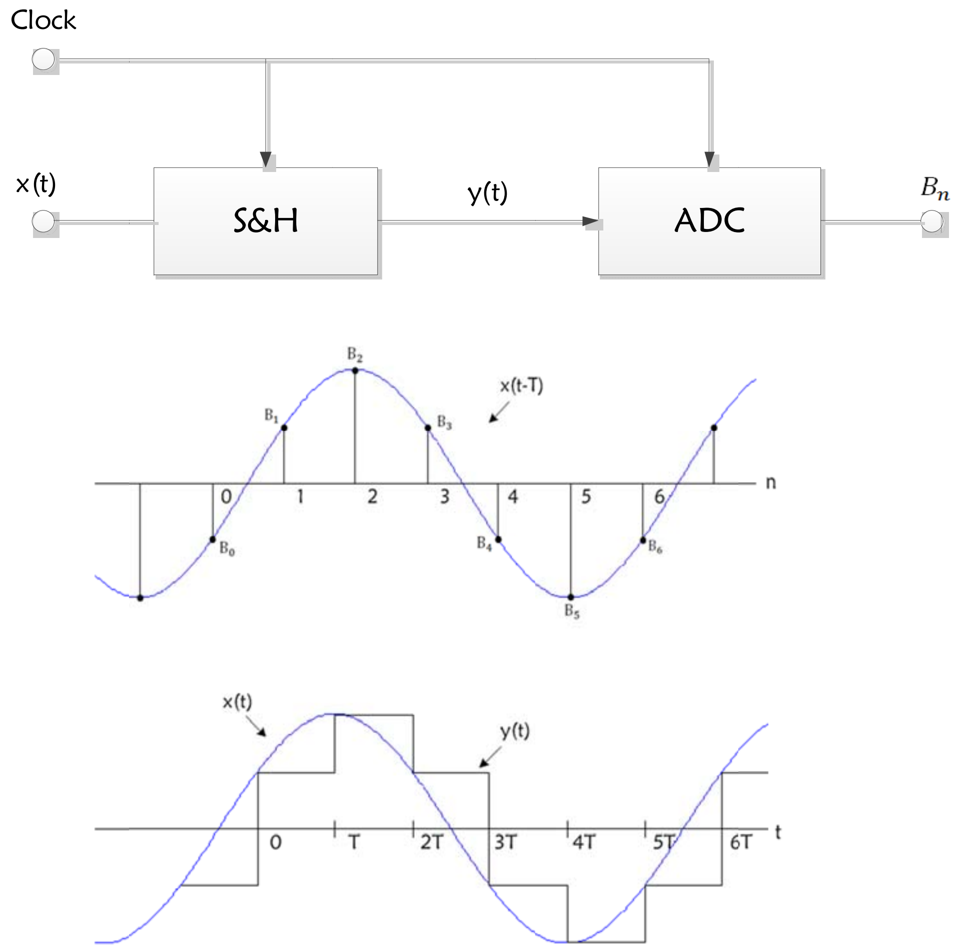


Figure 6-1 System for analog-to-digital conversion

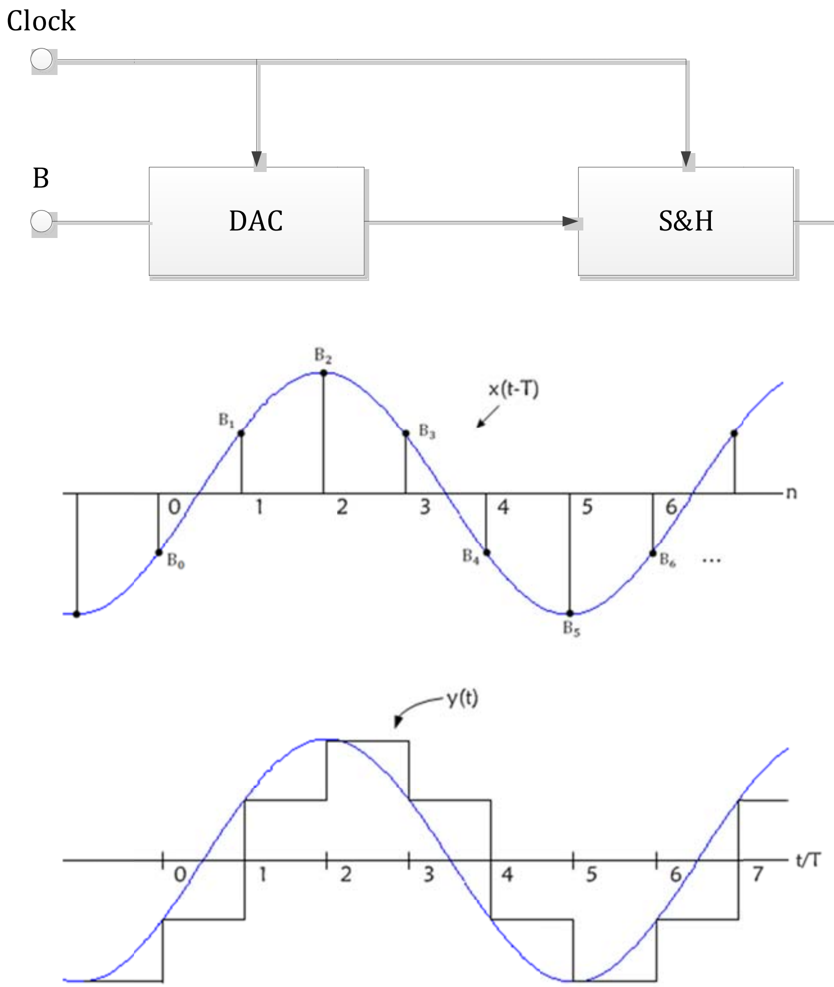


Figure 6-2 System for digital-to-analog conversion

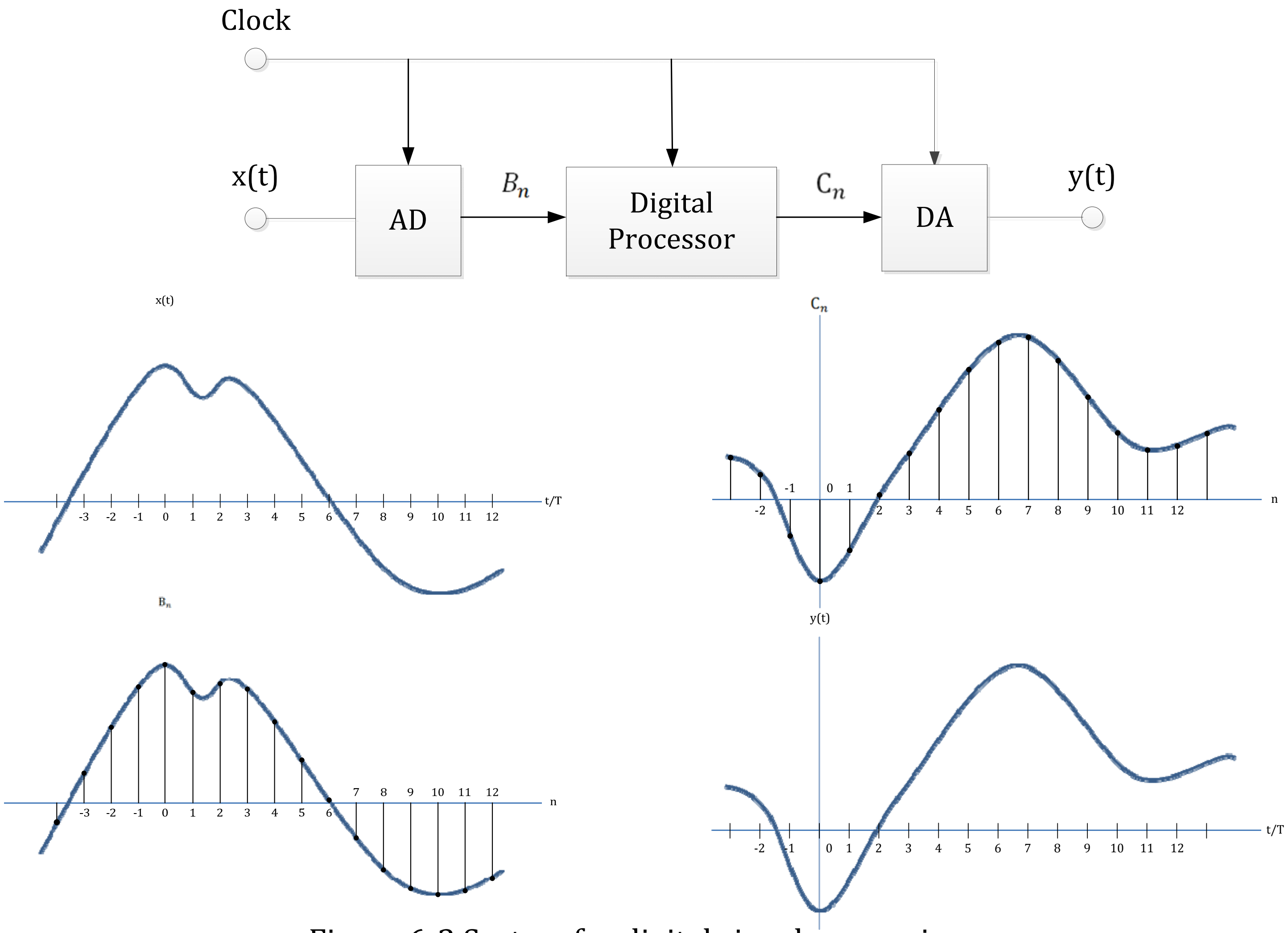


Figure 6-3 System for digital signal processing

6.2 Sampling

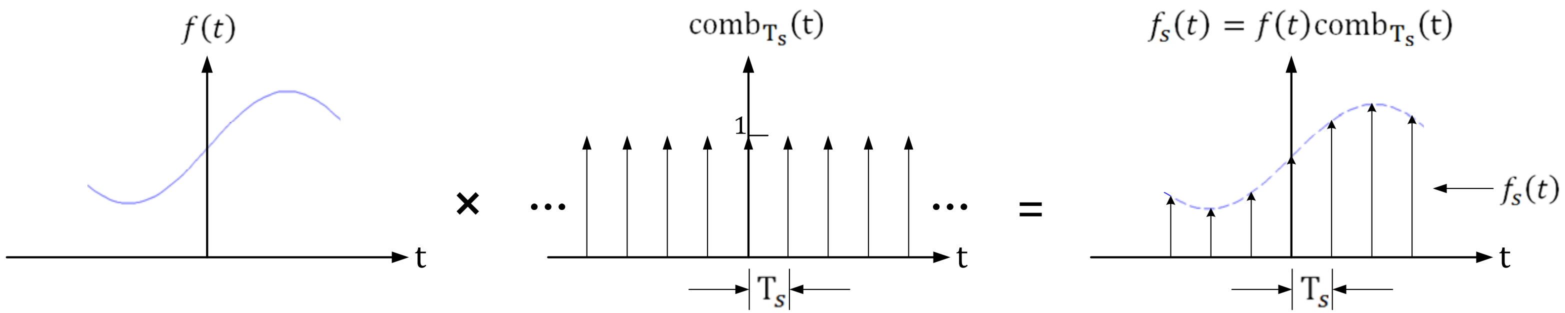


Figure 6-4

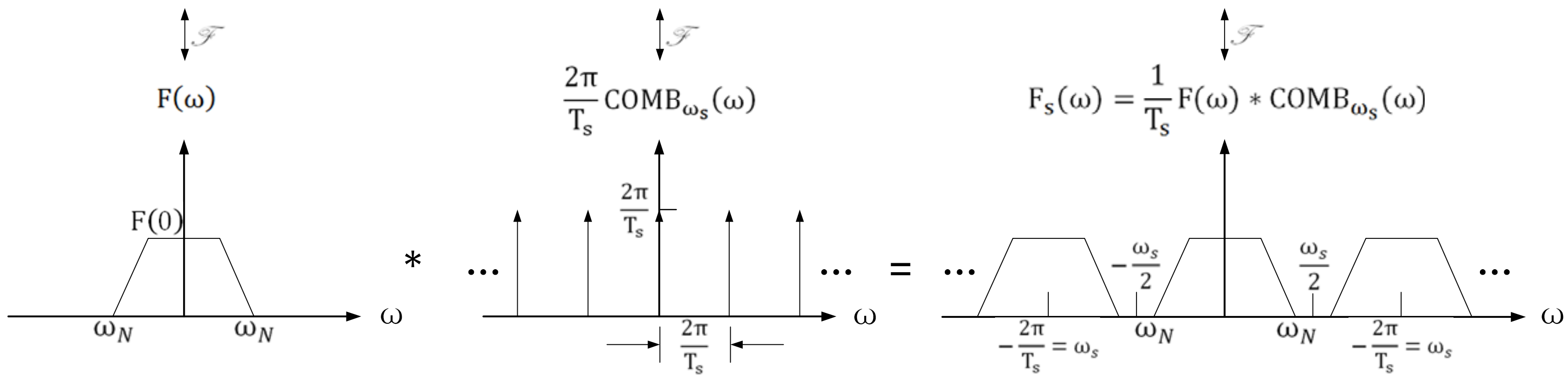


Figure 6-5

Sampling Period T_s

Sampling Rate (frequency) $\omega_s = \frac{2\pi}{T_s}$

$$f(t)\delta(t - nT) = f(nT_s)\delta(t - nT_s)$$

$$f_s(t) \triangleq f(t) \sum_{n=-\infty}^{\infty} \delta\,(t - nT_s)$$

$$= f(t)comb_{T_s}(t)$$

$$= \sum_{n=-\infty}^{\infty} f(nT_s)\,\delta(t - nT_s)$$

$$\Big\downarrow \quad \text{FT}$$

$$F_s(\omega) = \mathscr{F}\{f_s(t)\}$$

$$= \sum_{n=-\infty}^{\infty} f(nT_s)\mathscr{F}\{f_s(t)\}$$

$$= \sum_{n=-\infty}^{\infty} f(nT_s)\, e^{-jn\omega_s}$$

Poisson Sum Formula

$$\sum_{n=-\infty}^{\infty} f(t+nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}\, F(\omega_0) \qquad \omega_0 = \frac{2\pi}{T}$$

$$F_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega + n\omega_s)$$

Alternatively, we can show

$$F_s(w) = \mathscr{F}\{f(t)comb\} = \frac{1}{2\pi} F(\omega) * \mathscr{F}\{comb_{T_s}(t)\}$$

$$= \frac{1}{2\pi} F(\omega) * \frac{1}{2\pi} F(\omega) * [\frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)]$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega')\delta(\omega - n\omega_s - \omega')\, d\omega'$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} F(\omega + n\omega_s)$$

For a Positive-Time Signal

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(nT_s)\delta(t - nT_s)$$

$$F_s(\omega) = \frac{f(0^+)}{2} + \frac{1}{T_s}\sum_{n=-\infty}^{\infty} F(\omega + n\omega_s)$$

Examples: Find FT of sampled

$$\mathrm{f}(t) = e^{-|t|} \ \ and \ \ \mathrm{f}(t) = e^{-t}u(t)$$

Solution:

<1> $\mathrm{F}(\omega) = \mathscr{F}\{\mathrm{e}^{-|\mathrm{t}|}\}$

$$= \frac{2}{1 + w^2}$$

<2> $\mathrm{F}_s(\omega) = \mathscr{F}\{e^{-|t|} comb_{T_s}(t)\}$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} \frac{2}{1 + (\omega + n\omega_s)^2}$$

<3> $\mathrm{F}(\omega) = \mathscr{F}\{e^{-t}u(t)\}$

$$= \frac{1}{1 + j\omega}$$

<4> $\mathrm{F}_s(\omega) = \mathscr{F}\left\{e^{-t}u(t)comb_{T_s}(t)\right\}$

$$= \frac{1}{2} + \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\frac{2}{1 + j(\omega + n\omega_s)}$$

Notice that in

$$F_s(\omega) = \frac{1}{T_s} \sum F(\omega - n\omega_s)$$

The original $F(\omega)$ repeats every ω_s apart.

Let ω_N be upper frequency limit of $F(\omega)$ (i.e. $F(\omega) = 0 \; for \; \omega > \omega_N$).

If $\omega_s > 2\omega_N$, we see distinct $F(w\omega)$;

If $\omega_s < 2\omega_N$, we see $F(\omega)$ overlaps.

This spectral overlap is called "aliasing", to avoid aliasing, sample at rate $\omega_s > 2\omega_N$, or equivalently $T_s < \frac{T_N}{2} = \frac{1}{2f_N}$.

The critical values $\omega_s = 2\omega_N, T_s = \frac{1}{2f_N}$ are called Nyquist rate or interval.

If sampling rate is limited, then prefilter signal.

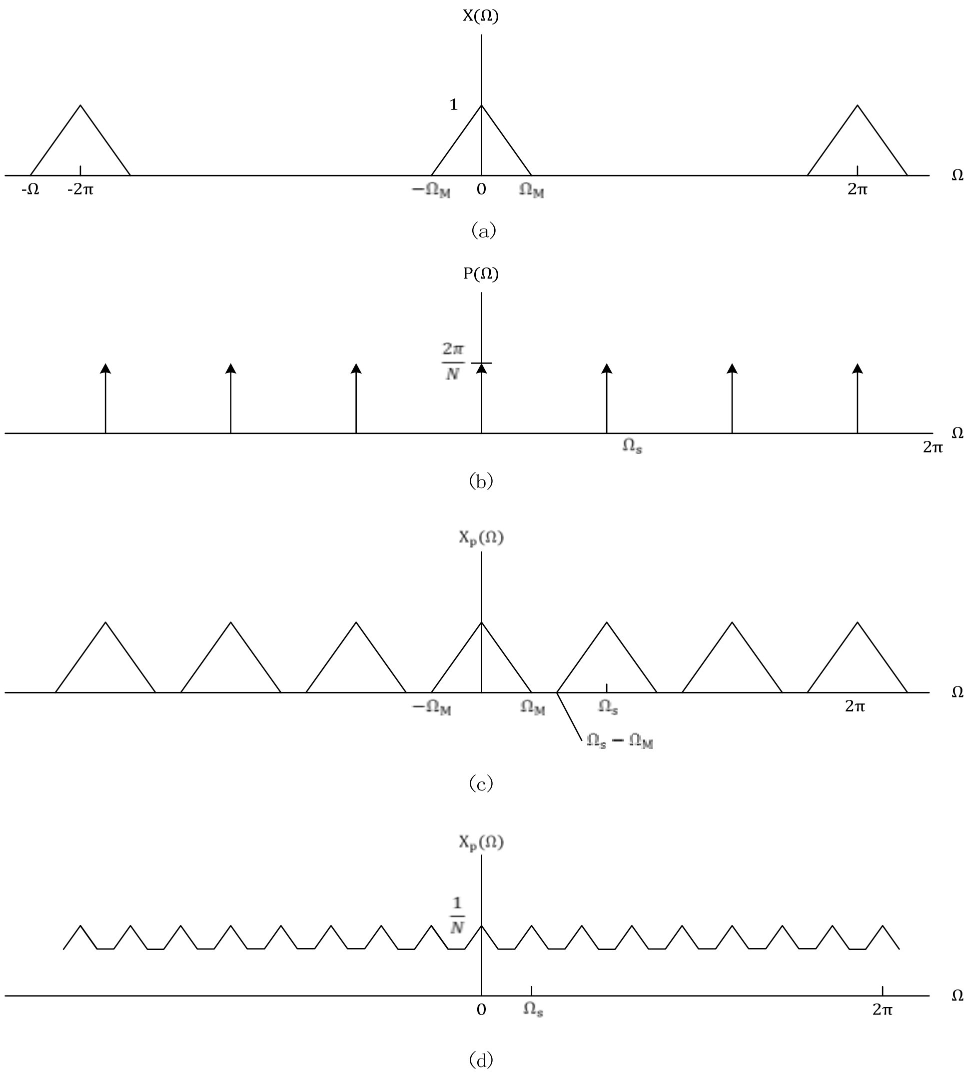


Figure 6-6

Effect in the frequency domain of impulse-train sampling of a discrete-time signal:
(a) spectrum of original signal;
(b) spectrum of sampling sequence;
(c) spectrum of sampled signal with $\Omega_s > 2\Omega_M$;
(d) spectrum of sampled signal with $\Omega_s < 2\Omega_M$.
Note that aliasing occurs.

6.3 Sampling Theorem

Theorem 1 —

A band-limited $(to\ w_N)$ signal $f(t)$ can be completed reconstructed from its sample values $f(nT_s)$ with

$$f(t) = \sum_{n=-\infty}^{\infty} T_s f(nT_s) \left\{ \frac{\sin\left[\frac{\omega_s(t-nT_s)}{2}\right]}{\pi(t-nT_s)} \right\}$$

$$\text{If } \omega_s \geq 2\omega_N$$

Proof:

$$F(\omega) = P_{\frac{\omega_s}{2}}(\omega) T_s F_s(\omega)$$

$$= P_{\frac{\omega_s}{2}}(\omega) T_s \sum_{n=-\infty}^{\infty} f(nT_s)\, e^{-jn\omega T_s}$$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\}$$

$$= \mathcal{F}^{-1}\left\{P_{\frac{\omega_s}{2}}(\omega) T_s \sum_{n=-\infty}^{\infty} f(nT_s)\, e^{-jn\omega T_s}\right\}$$

$$= T_s \sum_{n=-\infty}^{\infty} f(nT_s)\ \mathcal{F}\left\{P_{\frac{\omega_s}{2}}(\omega) e^{-jn\omega T_s}\right\}$$

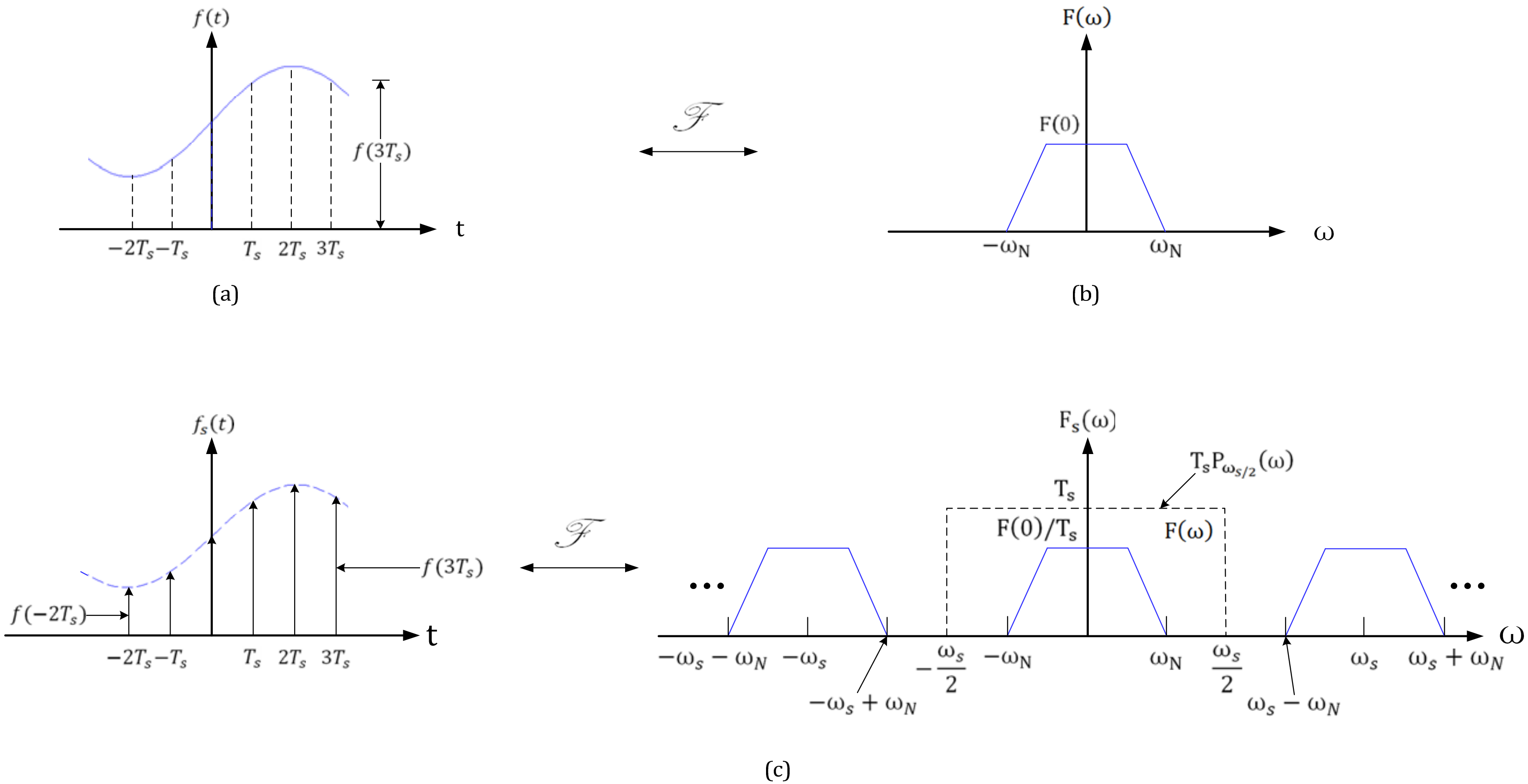


Figure 6-7 Illustrations of the Sampling Theorem

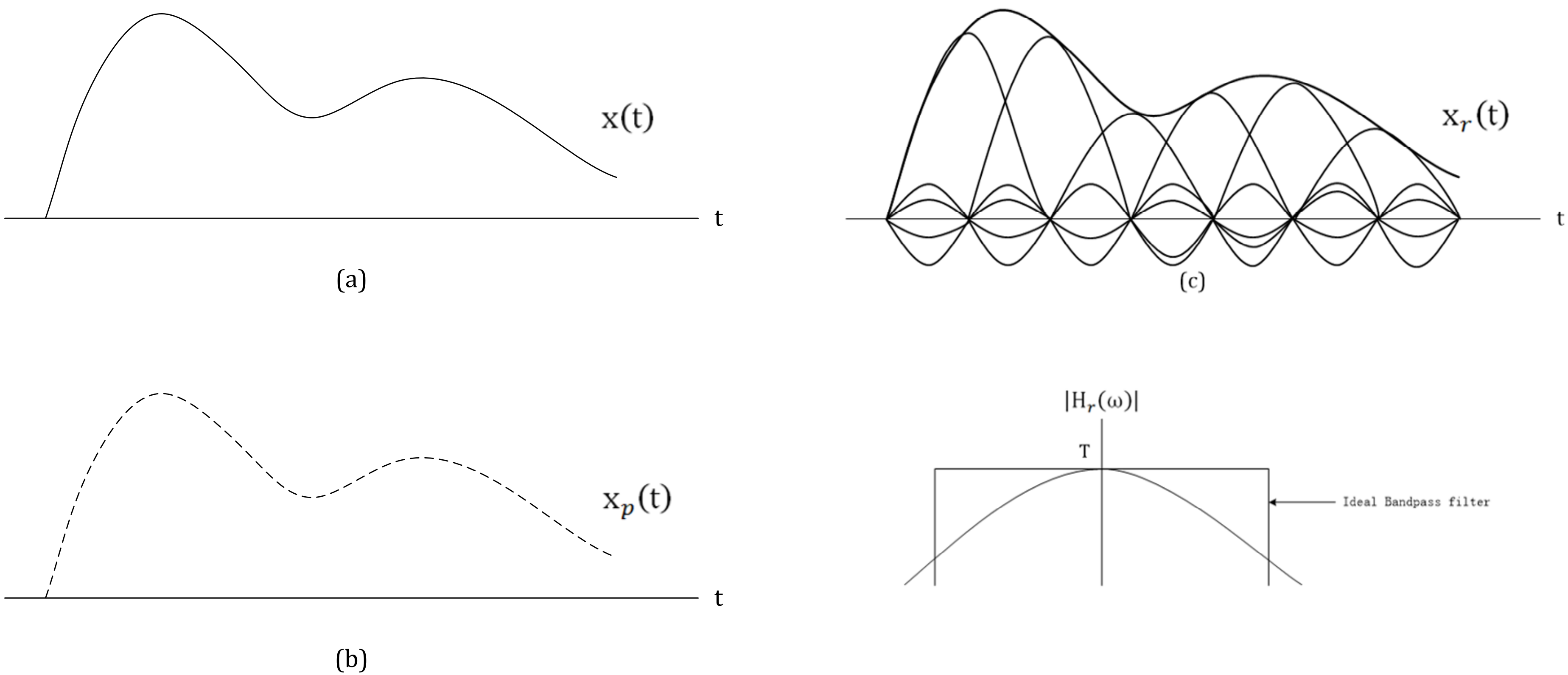


Figure 6-8

The signal $g(t) = 4\cos(200\pi t)\cos(1000\pi t)$, with spectrum shown in $Fig. 6 - 9\ (a)$, is sampled at (a) $f_s = 2kHz$ and (b) $f_s = 900Hz$. Sketch the sampled spectra for each sampling rate over the range $0 < f < f_s$ and determine if either of the sampled signals is aliased.

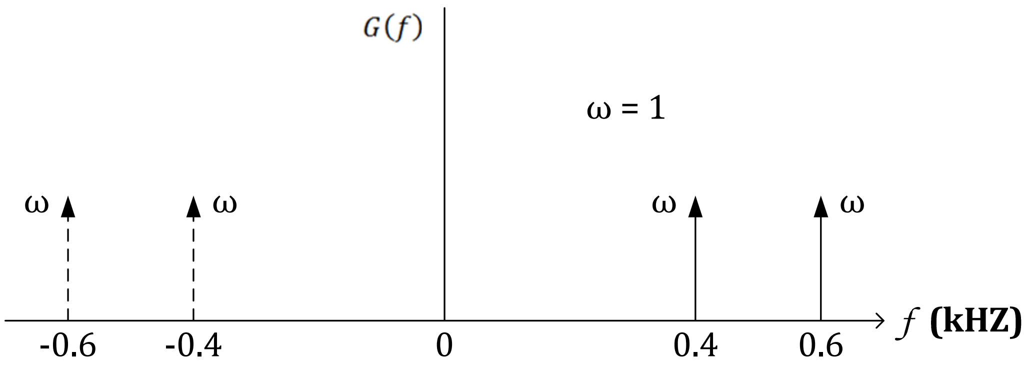


Fig. 6-9 (a) Cosine product

Solution: The first step is to determine the bandwidth (B) of $G(f)$, which from inspection of $Fig. 6 - 9\ (a)$ is $600\ Hz$. The corresponding Nyquist rate is $1.2 - kHz$; consequently, the $2\ kHz$ sampling rate is acceptable, whereas aliasing occurs at the $900 - Hz$ rate. The aliased spectrum shown in part (c) $(0 < f < 450)$ has a $300 - Hz$ component (dashed), which represents aliasing of the $600 - Hz$ cosine.

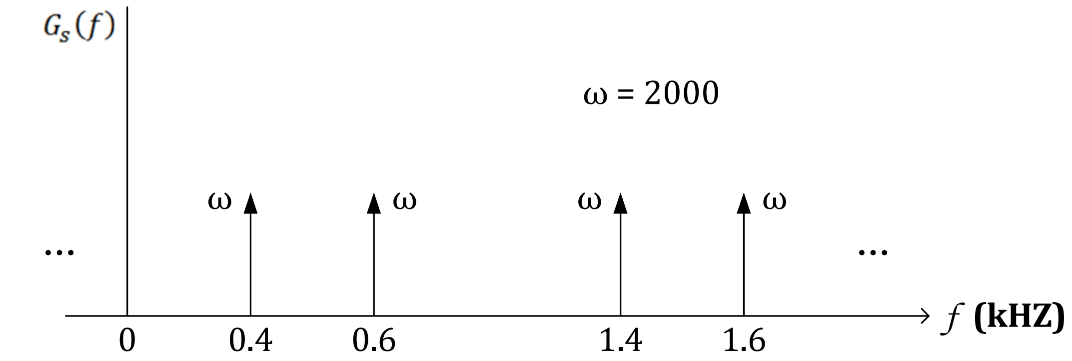


Fig. 6-9 (b) Sampling at 2 kHz

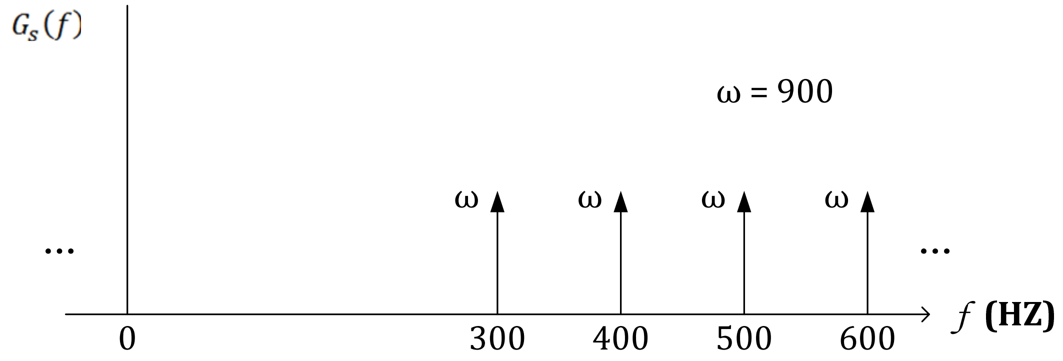


Fig. 6-9 (c) Sampling at 900 Hz

Theorem 2 — Frequency Sampling

A time-limited $(to\ T_N)$ signal $f(t)$

$$f(t) = 0 \ \ for \ \ |t| > T_N$$

Possesses a Fourier Transform that can be uniquely determined from its samples at frequencies $\frac{n\pi}{T_N}$ with

$$F(\omega) = \sum_{n=-\infty}^{\infty} F\left(\frac{n\pi}{T_N}\right) \frac{\sin(\omega T_N - n\pi)}{\omega T_N - n\pi}$$

If sampled at Nyquist rate.

Proof analogous to the Theorem 1

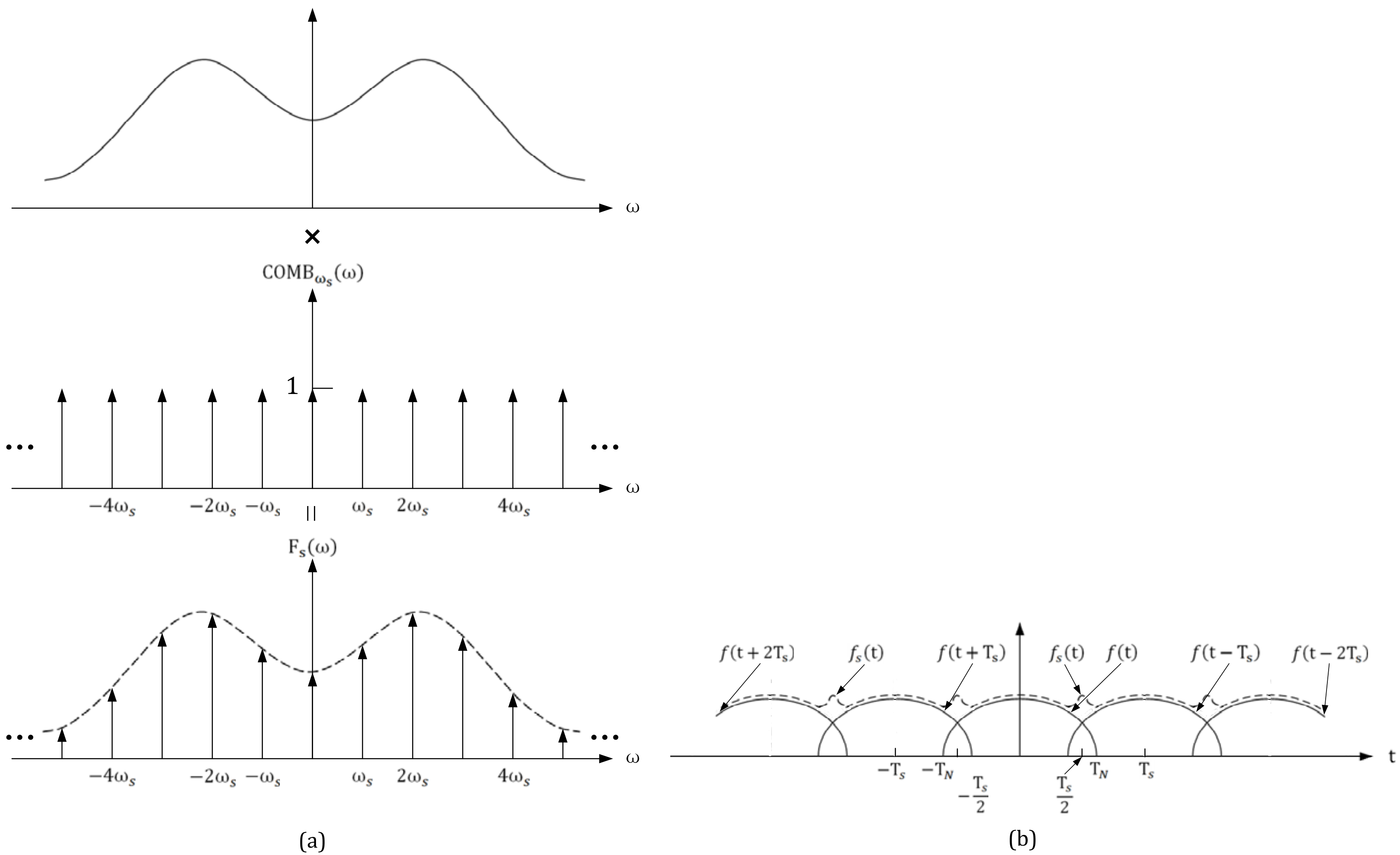


Figure 6-10 Illustrating the Time-Domain Aliasing Problem

Sampling with Rectangular Pulse Train — practical concerns:

$$f_p(t) = \sum_{n=-\infty}^{\infty} T_s \, P_{\frac{\omega_s}{2}}(t - nT_s)$$

$$F_p(\omega) = \sum_{n=-\infty}^{\infty} 2\pi \frac{\sin\left(\frac{n\omega_s \tau}{2}\right)}{\frac{n\omega_s \tau}{2}} \delta(\omega - n\omega_s)$$

$$f_s(t) = f(t) f_p(t)$$

$$F_s(\omega) = \mathrm{F} \left\{ f(t) f_p(t) \right\} = \frac{1}{2\pi} F(\omega) * F_p(\omega)$$

$$= \sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{n\omega_s \tau}{2}\right)}{\frac{n\omega_s \tau}{2}} \int_{-\infty}^{\infty} \delta(\omega' - n\omega) \, F(\omega - \omega') d\omega'$$

$$= \sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{n\omega_s \tau}{2}\right)}{\frac{n\omega_s \tau}{2}} F(\omega - n\omega_s)$$

Sinc function is caused by rectangle

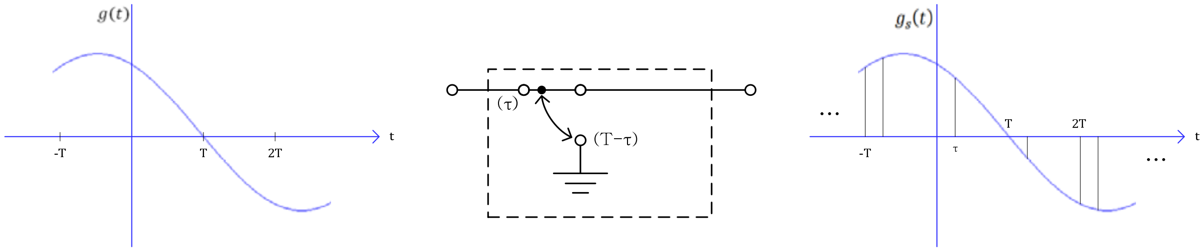
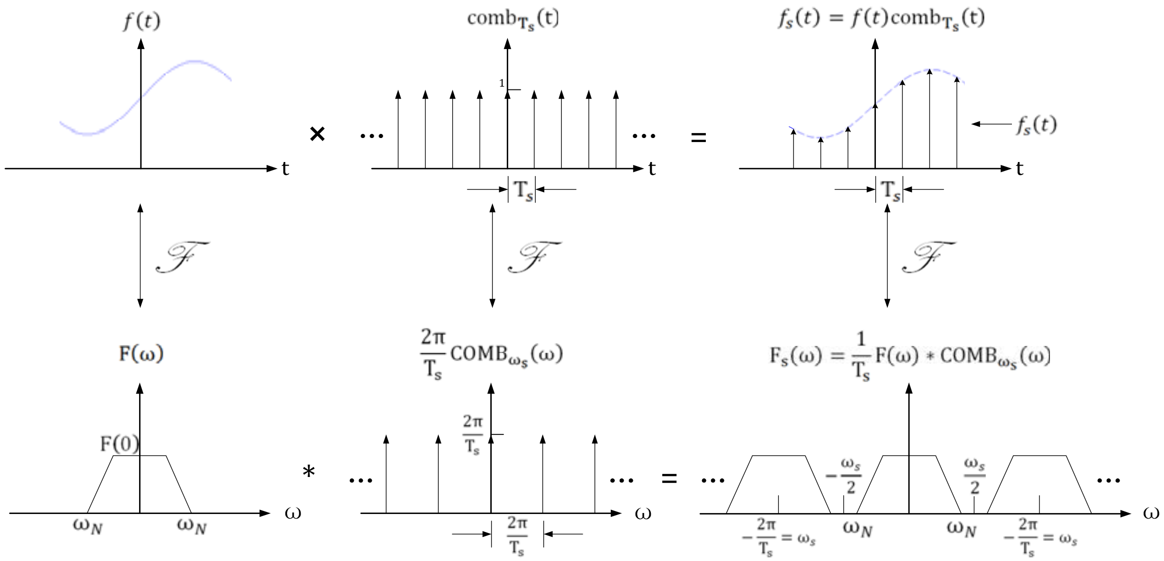


Figure 6-11 Finite-width pulse sampler



(a)

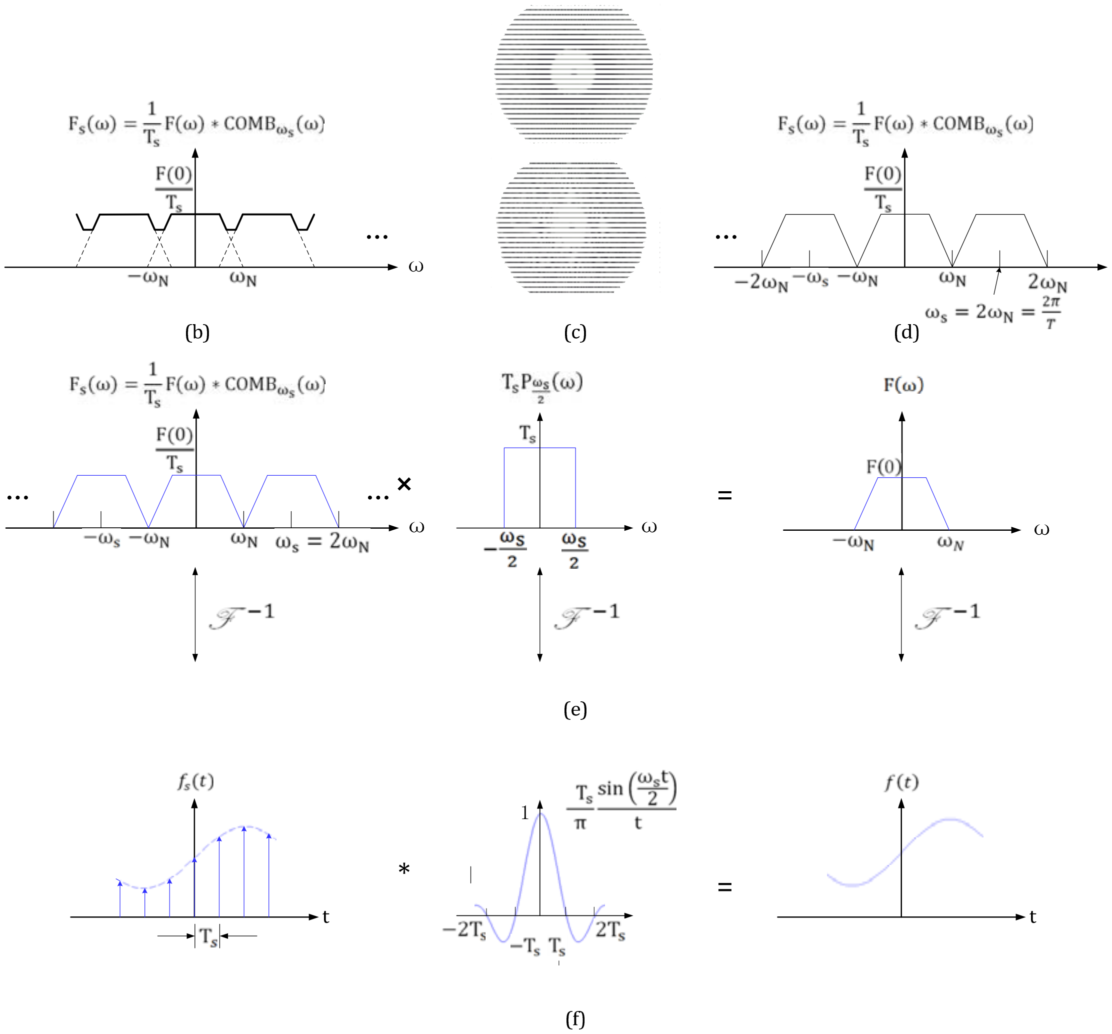
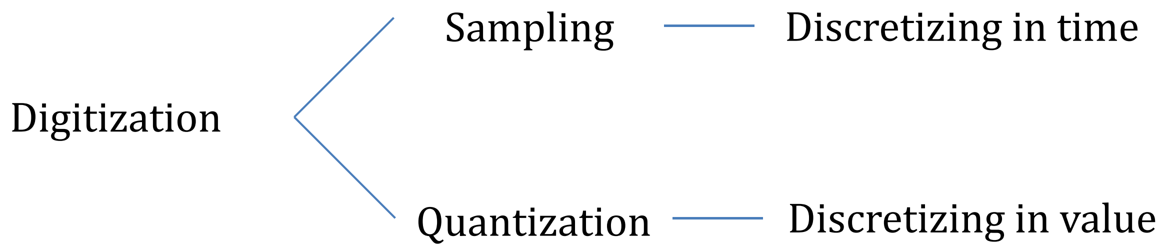
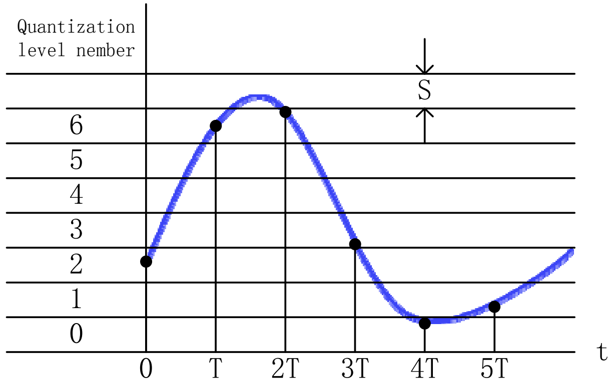


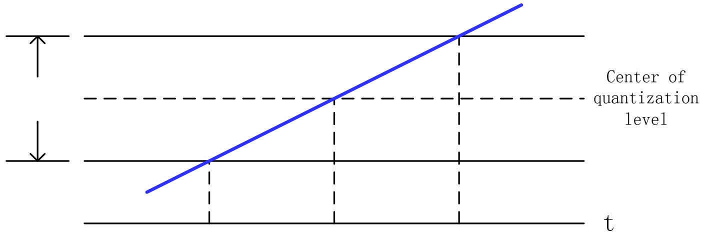
Figure 6-12

Delta sampling, Representation, and Recovery of Signals. [(c) Original image of spokewheel and its under-sample image. The effect of aliasing appears clearly. Reprinted by permission from Leger and Lee, Signal Processing Using Hybrid Systems.]

6.4 Quantization







(a) Quantizing

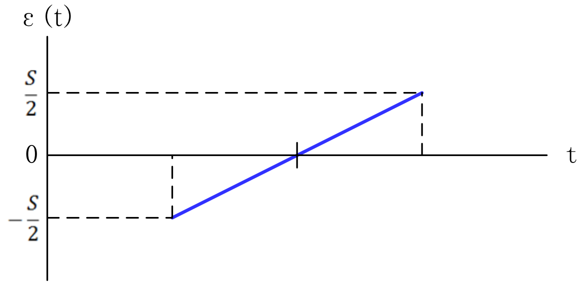


Figure 6-13

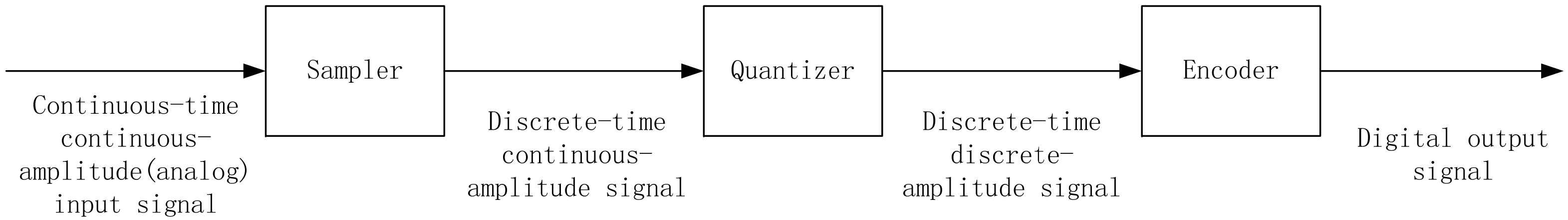
Mean-squared error:

$$\varepsilon(\alpha) = \frac{S}{2t_1} \bullet t$$

$$E = \frac{1}{2t_1}\int_{-t_1}^{t_1} \varepsilon^2(\alpha)\, d\alpha = \frac{1}{t_1}\int_{0}^{t_1} \varepsilon(\alpha)\, d\alpha$$

$$= \frac{1}{t_1}\int_{0}^{t_1} \left(\frac{S}{2t_1}\right)^2 \alpha^2 d\alpha = \frac{S^2}{12}$$

Sampling error can be avoided
by sampling at or above Nyquist rate.
Quantization error cannot be avoided.



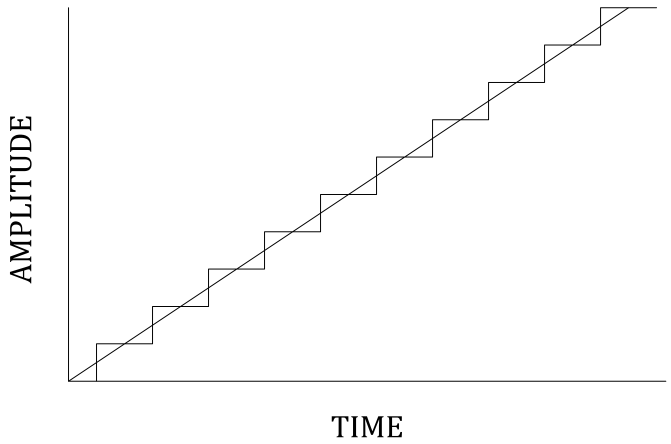


Figure 6-14-1 Digitized analog ramp

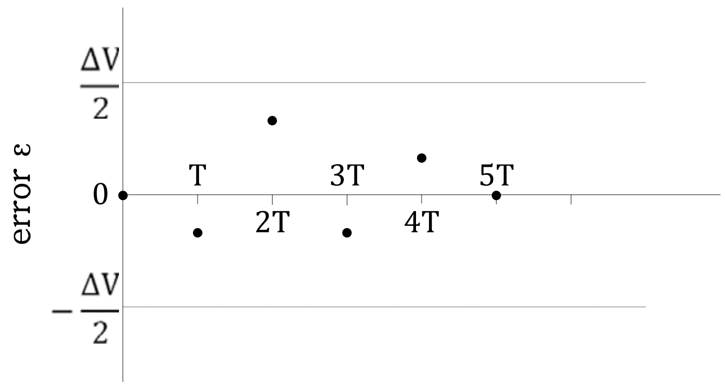


Figure 6-14-3 A/D converter errors vs. sample number

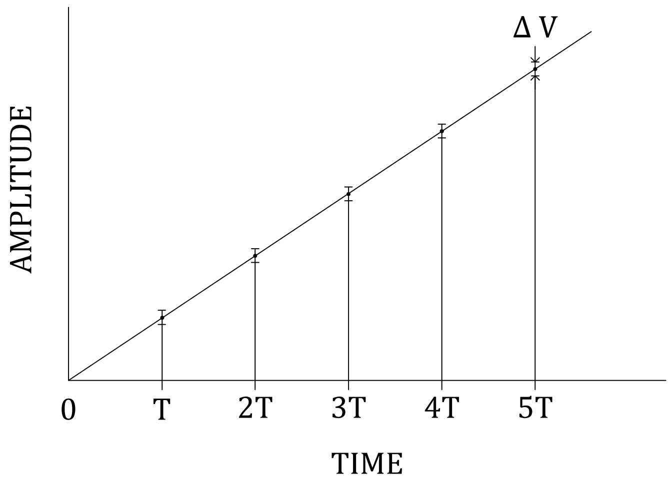


Figure 6-14-2 Digital signal resulting from ramp

6.5 Application in Telephone Transmission — Time-Division Multiplexing

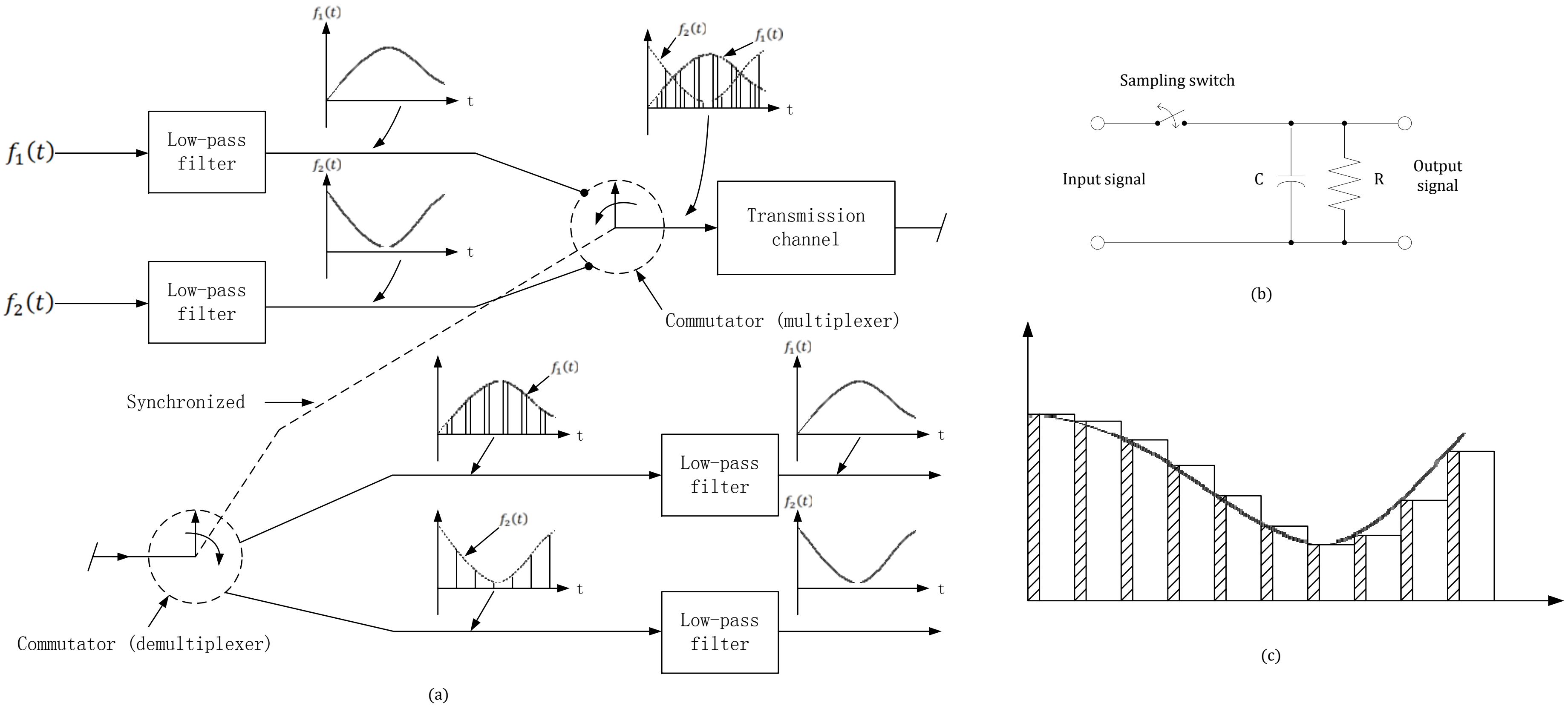


Figure 6-15 Time-Division Multiplexing